



Faculty of Graduate Studies  
Master Program in Mathematics

**General framework for managing the  
optimal strategies for an economic agent  
with uncertain lifetime**

By  
Salam Shrateh

Supervisor  
Dr. Abdelrahim S. Mousa

M.Sc.Thesis  
Birzeit University  
Palestine  
2022



Faculty of Graduate Studies  
Master Program in Mathematics

General framework for managing the  
optimal strategies for an economic agent  
with uncertain lifetime

By  
Salam Shrateh

This Thesis was defended successfully on February 1, 2022 and  
approved by

Committee Members:

Signature

Dr. Abdelrahim Mousa

(Head of Committee)

*Dr. Abdelrahim Mousa*

Dr Hassan Abu Hassan

(Internal Examiner)

*Dr. Hassan.. Abu Hassan*

Dr. Khalid Adarbeh

(External Examiner)

*[Signature]*

---

## الإهداء

إلى سندي في هذه الحياة  
أمي و أبي

---

# Acknowledgments

I would like to express my utmost honest gratitude to my supervisor, Dr. Abdelrahim Mousa for his effort in directing me to come with this Thesis. I also thanks Dr. Hassan Abu Hassan and Dr. Khalid Adarbeh for accepting to be committee members and also for the variable suggestions and comments.

# *Abstract*

In this Thesis we introduce an extension continuous time model for the famous Merton's model by considering the problem of an economic agent whose lifetime is uncertain and aims to determine the optimal strategies concerning social welfare purchase, life insurance purchase, consumption and investment. The economic agent aims to maximize an expected utility obtained from family consumption, size of estate in the event of premature death, and size of estate at retirement time if she lives that long. We assume that the economic agent agrees to sign contract with a selected (i) social welfare provider from an available social welfare market composed of finite number of welfare providers, and (ii) life insurance company from available life insurance market composed of finite number of life insurance providers. Meanwhile, first we assume that the economic agent invests all her saving in a financial market consists of one risky free asset and one risky asset. An extension to the problem under consideration is being formulated when the economic agent invests her saving in a financial market consists of finite number of risky assets. We restate the problem under consideration as an optimal control problem and apply the DPP to drive the corresponding HJB equation, a second order nonlinear PDE whose solution is the desired objective function. Finally, we manage to find explicit solution for the optimal strategies.

## المخلص

في هذه الرسالة نقدم نموذجًا زمنيًا ممتدًا مستمرًا لنموذج ميرتون الشهير من خلال النظر في مشكلة الوكيل الاقتصادي الذي تكون حياته غير مؤكدة ويهدف إلى تحديد الاستراتيجيات المثلى فيما يتعلق بشراء الرعاية الاجتماعية وشراء التأمين على الحياة والاستهلاك والاستثمار. يهدف الوكيل الاقتصادي إلى تعظيم المنفعة المتوقعة التي يتم الحصول عليها من استهلاك الأسرة ، وحجم التركة في حالة الوفاة المبكرة ، وحجم التركة في وقت التقاعد إذا كانت تعيش هذه المدة الطويلة. نفترض أن الوكيل الاقتصادي يوافق على توقيع عقد مع (١) مقدم رعاية اجتماعية مختار من سوق رعاية اجتماعية متاح يتألف من عدد محدود من مقدمي الرعاية الاجتماعية ، و (٢) شركة تأمين على الحياة من سوق التأمين على الحياة المتاح المكون من عدد محدود من مقدمو خدمات التأمين على الحياة. في غضون ذلك ، نفترض أولاً أن الوكيل الاقتصادي يستثمر كل مدخراته في سوق مالي يتكون من أصل واحد خال من المخاطر وأصل واحد محفوف بالمخاطر. تتم صياغة امتداد للمشكلة قيد النظر عندما يستثمر الوكيل الاقتصادي مدخراته في سوق مالي يتكون من عدد محدود من الأصول الخطرة. نعيد صياغة المشكلة قيد النظر باعتبارها مشكلة تحكم مثالية ونطبق DPP لدفع معادلة HJB المقابلة ، وهي DPP غير خطي من الدرجة الثانية وحلها هو الوظيفة الهدف المطلوبة. أخيرًا ، تمكنا من إيجاد حل واضح للاستراتيجيات المثلى.

---

# *Contents*

---

<b>List Of Figures</b>	<b>ix</b>
<b>List of symbols</b>	<b>x</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Basic preliminaries</b>	<b>5</b>
2.1 Brownian motion and Markov process . . . . .	13
2.2 Differential equations . . . . .	14
2.2.1 Ordinary differential equations . . . . .	15
2.2.2 Stochastic differential equations . . . . .	17
2.3 Itô's formula . . . . .	18
<b>3 Optimal strategies within a social market of L welfare providers and one risky asset</b>	<b>21</b>
3.1 Model setup . . . . .	22
3.1.1 Financial market with one risky asset . . . . .	22
3.1.2 Life insurance market . . . . .	23
3.1.3 Social welfare market . . . . .	25
3.1.4 Wealth process . . . . .	28
3.2 Stochastic optimal control problem . . . . .	30
3.3 Hamiltonian theorem . . . . .	33
3.4 Optimal strategies in terms of the value function . . . . .	38
3.5 The family of discounted constant relative risk aversion utilities . . . . .	45
3.6 Explicit solution . . . . .	45
<b>4 Generalized form: optimal strategies within a social market of L welfare providers and financial market of N risky assets</b>	<b>57</b>
4.1 General framework . . . . .	57
4.1.1 Generalized industrial market models . . . . .	58
4.1.1.1 Financial market with N risky assets . . . . .	58
4.1.2 Generalized wealth process . . . . .	60
4.2 Generalized stochastic optimal control problem . . . . .	61
4.3 Dynamic programming principle . . . . .	62

---

4.4	Generalized optimal strategies in terms of the value function . . . . .	63
4.5	Explicit solution for the generalized optimal strategies . . . . .	68
<b>5</b>	<b>Conclusion</b>	<b>79</b>
	References . . . . .	80



---

## *List of Figures*

---

2.1	A SAMPLE PATH FOR A SOLUTION OF A DIFFERENTIAL EQUATION . . . .	15
2.2	A SAMPLE PATH FOR A SOLUTION OF A STOCHASTIC DIFFERENTIAL EQUATION . . . . .	17

---

## *List of symbols*

---

$\mathcal{B}$	Borel $\sigma$ -algebra
$\mathbb{N}$	Natural numbers
$\mathbb{R}$	Real numbers
RV	Random variable
$\mathbb{R}^n$	Space of $n$ -dimension
$\mathcal{F}$	Sigma algebra
$\mathcal{F}_t$	Filtration
$\emptyset$	Empty set
inf	Infimum
sup	Supremum
min	Minimum
max	Maximum
$\otimes$	Tensor product
$f^{-1}$	Inverse image
HJB	Hamilton-Jacobi-Bellman
ODE	Ordinary differential equation
PDE	Partial differential equation
SDE	Stochastic differential equation
BM	Brownian motion
DPP	Dynamic programming principle
$\mathbb{E}$	Expected value
CRRA	Constant relative risk aversion
FM	Financial market
w.r.t	With respect to
$T \wedge \tau =$	$\min\{T, \tau\}$
OCP	Optimal control problem
$\forall$	For all
$\exists$	There exists
$\in$	Belong to

# *Chapter 1*

---

## *Introduction*

---

In this Thesis we extend the work done by Moath in his reference [20] in the following sense: (i) Mousa et al. in [25] introduced a problem of obtaining the optimal strategies for the wage-earner with an uncertain lifetime. In the final Chapter, they got the solution in the case of constant relative risk aversion (CRRA) utilities. (ii) Sondos [21] studied the problem of finding the optimal strategies for an economic-agent whose lifetime is uncertain and enters the social security system. In special case of discounted (CRRA) utility function she got the optimal strategies of consumption, investment and life insurance selection within registering in the social welfare system but the welfare level was not being controlled. (iii) Moath [20] made an extension on Sondos' work. He found the optimal strategies for a wage earner with uncertain lifetime where the welfare policy is being a new control variable in the case where the social welfare system consist of only on welfare provider. He used the dynamic programming principle derive Hamilton-Jacobi-Bellman (HJB) equation and then try to find explicit solution using a special case of family which is CRRA.

We make an extension by allowing the welfare policy to be an additional control

---

variable in the case where the social welfare system consist of finite number of welfare providers. We assume that the economic agent has access the welfare policy as another kind of family protection. To address this problem, we're going to use the idea of dynamic programming principle. We will derive the result which is nonlinear partial differential equation whose solution is the value function of the problem under consideration using dynamic programming principle (DPP).

The starting point and development of Dynamic Programming Principle (DPP) was attributed to Bellman's work in [5, 6]. In [7], he made an extension on his previous work with deterministic control process. As we know, the solution of deterministic control process is a backwards recursive relation which is a partial differential equation of order two whose solution is the desired value function. More details were discussed in [14, 15, 39].

Yarri was the first researcher who started working in the filed of optimal financial planing decision under uncertain lifetime in his reference [34]. After that, many papers have been published based on Yarri's research such as the work done by Hakansson in [17, 18]. He worked in a discrete - lifetime and his purpose was to maximize the expected utility for the entrepreneur from investing in money and financial markets.

Similarly, in a discrete-time framework, Fischer in his research [13] examined life-cycle patterns of optimal insurance purchase in detail using the dynamic programming technique and obtained the formula for the present value of the future income, a formula that is different from the one under a certain lifetime. He emphasis in his paper on the comparative static and dynamics of the insurance demand and function, more than existence of a solution itself.

The stochastic model of optimal investment and consumption decision was introduced by Merton in [23, 24]. His models studied an individual with a fixed income who aspires to maximize the utility return of consumption and wealth.

---

Richard collected between Yarri's model and Merton's programming in his reference in [32]. Richard in [32] and Fischer in [13] both got a human capital theorem, but Fischer got it only for function with Constant relative risk aversion (CRRA) utilities. More realistic models are still developing today. For example, Andersen Model [3] and Markove modulated risk model [4].

In 1991, Dumas in [10] studied a problem for a investor who saved his wealth without consumption. At final point, the whole wealth was consumed.

A stochastic optimal linear quadratic control problem faced in [2], while in [40] there is an additional condition that is forbidden to have short selling of stocks. In, [26], H. Ou-Yang provided a model which had an interaction between investor and professional manger.

Pliska and Ye in 2007 in their reference [27] depend on their work by Ye [35, 36]. They considered a problem notable by a wage earner with continuous lifetime related with finite number of risky assets. In addition, Pliska and Ye in [27] extended on Merton's model for a wage earner who have a random time in life.

Bellman's principle of optimality plays an important role with applications, hence many papers used it later such as [2] in 2010.

Duarte et al. in [9] expanded Merton's model. In addition, they relied on reference [27], but with a financial market (FM) comprised of one risk-free and an arbitrary number of risky securities in multidimensional Brownian motion (BM) space. They discussed some properties of the solution in the case of CRRA utilities.

Shenab and Weib in [30] deemed a model for a wage earner in a complete market with BM and unbounded parameter.

H. M. Soner and N. Touz based on Merton problem they consider a problem with small proportional transaction costs in their reference [32]. They used the asymptotic analysis to approach the solution of their problem.

---

In 2019, Ye released new papers including stochastic problems. For example, in [38] he used the marginal approach in order to solve his problem and achieved his goal to maximize the expected utility. A problem of random distribution kernels which defined on a product space discussed in [37].

In this research, we will extend the work done by Mousa et al. in [25] and Moath [20] by allowing the welfare policy to be a control variable in the economic agent's problem. We assume that the economic agent has access the welfare policy as another kind of family protection through a contracts which starts at  $t \in [0, T]$ , where  $T$  is the retirement age for the economic agent. The new setting in this paper is that the economic agent has access to  $L$  welfare providers and want to choose one only at a time. We consider that the economic agent is paying an amount  $q_l(t)$ , where  $l = 1, 2, \dots, L$  to the  $l$  social welfare which is called the welfare premium. The welfare premium  $q_l(t)$  is assumed to be based on a welfare rate  $h_l(t)$  which is determined by the social welfare provider itself. If the economic agent perish at time  $\tau < T$  while participating in the social welfare policy, the social welfare pays a sum of money given by

$$\frac{q_l(\tau)}{h_l(\tau)},$$

as a substitution to the family.

We organize this Thesis as follows. In Chapter 2 we will review some definitions which help us in our thesis. In Chapter 3, we will look for the optimal strategies under a social market of  $L$  welfare providers and one risky asset. To reach such strategies we use dynamic programming principle technique. In Section 3.5, we reach an important proposition in the case of CRRA utilities. In Chapter 4 we use similar technique in order to study the problem of the economic agent who enter industrial market with finite number of risky security assets. In the final Chapter we get our conclusion.

Our main results in this Thesis are Theorem 4.2 and Proposition 4.1. Note that the readers suppose to have a good knowledge in measure theory.

## Chapter 2

---

### *Basic preliminaries*

---

The purpose of chapter 2 is to review some definitions that will be needed through this thesis.

**Definition 2.1.** [12] Let  $\Omega$  be a nonempty set. A  $\sigma$ -algebra on  $\Omega$  is a collection  $\mathcal{F}$  of subsets of  $\Omega$  with these properties:

1.  $\emptyset, \Omega \in \mathcal{F}$ .
2. If  $B \in \mathcal{F}$ , then  $B^c \in \mathcal{F}$ .
3. If  $B_1, B_2, \dots \in \mathcal{F}$ , then

$$\bigcup_{i=1}^{\infty} B_i \in \mathcal{F}.$$

**Example 2.1.** Let  $\Omega = \mathbb{N}$ . Then

$$\mathcal{F} = \{\emptyset, \Omega, \{1, 3, 5, \dots\}, \{2, 4, 6, \dots\}\},$$

is  $\sigma$ -algebra, where  $\mathbb{N}$  is the natural numbers.

**Definition 2.2.** [12] If  $\mathcal{F}$  is a  $\sigma$ -algebra on  $\Omega$  we called  $(\Omega, \mathcal{F})$  a measurable space.

---

**Example 2.2.** Back to example 2.1  $(\Omega, \mathcal{F})$  satisfies 2.2.

**Definition 2.3.** [12] We name

$$\mathbb{P} : \mathcal{F} \rightarrow [0, 1],$$

a probability measure if:

(1)  $\mathbb{P}(\emptyset) = 0.$

(2)  $\mathbb{P}(\Omega) = 1.$

(3)  $0 \leq \mathbb{P}(B) \leq 1,$  for all  $B \in \mathcal{F}$

(4)  $\mathbb{P}(\bigcup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(B_i),$  If  $B_1, B_2, \dots \in \mathcal{F}.$

(5) If  $B_1, B_2, \dots$  are disjoint sets in  $\mathcal{F},$  implies

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(B_i).$$

(6) if  $C, D \in \mathcal{F},$  then

$$C \subseteq D \text{ implies } \mathbb{P}(C) \leq \mathbb{P}(D) .$$

**Definition 2.4.** [12] A triple  $(\Omega, \mathcal{F}, \mathbb{P})$  is called a probability space.

**Remark 2.1.** [12] A property that's true exclude for an event of probability 0 is called to occur almost surely ( abbreviated "a.s." ).

**Definition 2.5.** [12]  $(\Omega, \mathcal{F}, \mathbb{P})$  is named a complete probability space if  $\forall A \in \mathcal{F}$  with  $\mathbb{P}(A) = 0, \forall G \subseteq A$  then  $G \in \mathcal{F}.$

**Definition 2.6.** [29] Let  $\mathcal{C} \subseteq \Omega,$  then there is a smallest  $\sigma$  - algebra  $\mathcal{H}_{\mathcal{C}}$  consisting  $\mathcal{C},$  meant

$$\mathcal{H}_{\mathcal{C}} = \bigcap \{ \mathcal{H}, \mathcal{H} \text{ is } \sigma - \text{ algebra of } \Omega, \mathcal{C} \subseteq \mathcal{H} \},$$

or named the  $\sigma$  - algebra generated by  $\mathcal{C}.$

**Definition 2.7.** [29] If  $\mathcal{C} \subseteq \mathbb{R}^n,$  then the Borel  $\sigma$  - algebra  $\mathcal{B}$  is the smallest  $\sigma$  - algebra of subsets of  $\mathbb{R}^n$  consisting all open sets.



---

**Definition 2.8.** [28] *The collection of Lebesgue measurable sets is a  $\sigma$ -algebra which contains all open sets and all closed sets.*

**Definition 2.9.** [22] *A mapping*

$$Y : \Omega \rightarrow \mathbb{R}^n,$$

*defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  is named  $\mathcal{F}$ -measurable if*

$$Y^{-1}(U) := \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F},$$

*$\forall$  open sets  $U \in \mathbb{R}^n$ .*

**Definition 2.10.** [12] *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. A function*

$$\mathbf{Y} : \Omega \rightarrow \mathbb{R}^n,$$

*is named an  $n$ -dimensional random variable (RV) if  $\forall B \in \mathcal{B}$ , we get*

$$\mathbf{Y}^{-1}(B) \in \mathcal{F},$$

*where  $\mathcal{B}$  denotes the collection of Borel subsets of  $\mathbb{R}^n$ , which is the smallest  $\sigma$ -algebra of subsets of  $\mathbb{R}^n$  containing all open sets and  $B$  are called Borel sets.*

**Definition 2.11.** [12] *Let  $A \in \mathcal{F}$ : Then*

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases} \quad (2.1)$$

*is called the indicator function of  $A$ .*

**Example 2.3.** *The indicator function of  $A$  is a RV.*

**Definition 2.12.** [22] *A collection of RV  $X_t, t \in T$  with values in  $\mathbb{R}$  is named a stochastic process.*

**Definition 2.13.** [8] *Let  $(X_t)_{t \in \mathbb{R}_+}$  be a stochastic process on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and valued on the measurable space  $(E, \mathcal{B})$  (i.e.,  $E$  is  $\sigma$ -algebra on  $\mathcal{B}$ ).*

---

The process  $(X_t)_{t \in \mathbb{R}_+}$  is measurable if it is measurable as a mapping defined from

$$\mathbb{R}_+ \times \Omega \rightarrow E.$$

**Definition 2.14.** [29] A filtration is a family

$$\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0},$$

of  $\sigma$ -algebras verifies

$$0 \leq r < t \Rightarrow \mathcal{F}_r \subset \mathcal{F}_t.$$

**Definition 2.15.** [12] A stochastic process

$$X = \{x_t : t \in T\},$$

on a probability space is adapted to the filtration if for any  $t$ ,  $X_t$  is an  $\mathcal{F}_t$ -measurable RV.

**Definition 2.16.** [8] The process  $(X_t)_{t \in \mathbb{R}_+}$  is called progressively measurable relative to the filtration  $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$  if, for all  $t \in \mathbb{R}_+$ , the function

$$(s, \omega) \in [0, t] \times \Omega \rightarrow X(s, \omega),$$

is  $(\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t)$ -measurable.

**Definition 2.17.** [8] A real-valued process is named predictable relative to a filtration  $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ , if a mapping

$$\mathbb{R}_+ \times \Omega \rightarrow \mathbb{R},$$

is measurable relative to the  $\sigma$ -algebra generated by the same filtration.

**Definition 2.18.** [29] If

$$\int_{\Omega} |X(\omega)| d\mathbb{P}(\omega) < \infty,$$

---

then

$$E[X] := \int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \int_{\mathbf{R}^n} x d\mu_X(x),$$

is named the expectation of  $X$ .

**Definition 2.19.** [29] Two subsets  $B, C \in \mathcal{F}$  are independent if

$$\mathbb{P}(C \cap B) = \mathbb{P}(C)\mathbb{P}(B).$$

**Definition 2.20.** [29] If two RV  $X, Y$  are independent then

$$E[XY] = E[X]E[Y],$$

needed  $E[|X|], E[|Y|] < \infty$ .

**Definition 2.21.** [12]

$$V(\mathbf{X}) := \int_{\Omega} |\mathbf{X} - E(\mathbf{X})|^2 d\mathbb{P},$$

called the variance of  $\mathbf{X}$ .

**Definition 2.22.** [12] The distribution function of  $\mathbf{X}$  is

$$F_{\mathbf{X}} : \mathbb{R}^n \rightarrow [0, 1],$$

and defined by

$$F_{\mathbf{X}}(x) := \mathbb{P}(\mathbf{X} \leq x), \quad \forall x \in \mathbb{R}^n.$$

**Definition 2.23.** [8] Let  $X$  be a RV. The mapping

$$F_X : \mathbb{R} \rightarrow [0, 1],$$

with

$$F_X(t) = \mathbb{P}([X \leq t]), \quad \forall t \in \mathbb{R}.$$

Named the cumulative distribution function of  $X$ .

---

**Definition 2.24.** [12] *The joint distribution function*

$$F_{\mathbf{Y}_1, \dots, \mathbf{Y}_m} : (\mathbb{R}^n)^m \rightarrow [0, 1],$$

of the RV

$$\mathbf{Y}_1, \dots, \mathbf{Y}_m : \Omega \rightarrow \mathbb{R}^n,$$

is

$$F_{\mathbf{Y}_1, \dots, \mathbf{Y}_m}(y_1, \dots, y_m) := \mathbb{P}(\mathbf{Y}_1 \leq y_1, \dots, \mathbf{Y}_m \leq y_m)$$

$$\forall y_k \in \mathbb{R}^n, k = 1, \dots, m.$$

**Definition 2.25.** [12] *Let*

$$\mathbf{X} : \Omega \rightarrow \mathbb{R}^n,$$

is a RV and  $F$  its distribution function. If  $\exists$  a nonnegative, integrable function

$$f : \mathbb{R}^n \rightarrow \mathbb{R},$$

such that

$$F(x) = F(x_1, \dots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y_1, \dots, y_n) dy_n \cdots dy_1.$$

Hence  $f$  is named the density function.

**Theorem 2.1.** [33] *Define a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $X$  and  $Y$  integrable RV on  $\Omega$ , and  $\mathcal{C}$  and  $\mathcal{B}$  sub- $\sigma$  fields of  $\mathcal{F}$ .*

(i)  $E[\beta X + \beta Y | \mathcal{C}] = \beta E[X | \mathcal{C}] + \beta E[Y | \mathcal{C}]$  for  $\beta, \beta \in \mathbf{R}$ .

(ii) If  $X \geq Y$  then  $E[X | \mathcal{C}] \geq E[Y | \mathcal{C}]$ .

**Definition 2.26.** [29] *An  $n$ -dimensional stochastic process  $\{M_t\}_{t \geq 0}$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  is named a martingale to  $\{\mathcal{M}_t\}_{t \geq 0}$  if*

(i)  $M_t$  is  $\mathcal{M}_t$ -measurable for all  $t$ ,

---

(ii)  $E [|M_t|] < \infty \forall t$ .

(iii)  $E [M_s | \mathcal{M}_t] = M_t \forall s \geq t$ .

**Definition 2.27.** [12] Suppose

$$g : [0, T] \rightarrow \mathbb{R},$$

is continuously differentiable deterministic function and not a stochastic process, with  $g(0) = g(T) = 0$ , then we define

$$\int_0^T g dW = - \int_0^T g' W dt,$$

where  $\int_0^T g dW$  is a RV.

**Lemma 2.2.** [12] Let  $g$  be a function that satisfies the previous definition, then

$$\mathbb{E} \left[ \int_0^T g dW \right] = 0.$$

**Definition 2.28.** [12] A Gaussian process is a stochastic process  $\{X_t; t \in T\}$  for which any finite linear combination of samples will be normally distributed.

**Theorem 2.1. Fubini -Tonelli theorem** [11]

Suppose  $A$  and  $B$  are  $\sigma$ -finite measure spaces. not necessarily complete, and if either

$$\int_A \left( \int_B f(x, y) dy \right) dx < \infty,$$

or

$$\int_B \left( \int_A f(x, y) dx \right) dy < \infty,$$

then

$$\int_{A \times B} |f(x, y)| d(x, y) < \infty,$$

and

$$\int_A \left( \int_B f(x, y) dy \right) dx = \int_B \left( \int_A f(x, y) dx \right) dy = \int_{A \times B} f(x, y) d(x, y).$$

---

**Definition 2.29.** [16] Assume function  $f(x, y)$  is a smooth function. The gradient of  $f(x, y)$  at a point  $(x_0, y_0)$  is the vector

$$\nabla f(x_0, y_0) = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}.$$

**Definition 2.30.** [19]

Given the optimization problem

$$\max f(x),$$

subject to

$$g_i(x) \leq 0, \quad \forall i \in I,$$

$$h_j(x) = 0, \quad \forall j \in J,$$

Then  $x^*$  a local maximum if there exists multipliers  $\mu_i \geq 0, \quad \forall i \in I$  and  $\lambda_j$  for all  $j \in J$  such that

$$\begin{aligned} \nabla f(x^*) - \sum_i \mu_i \nabla g_i(x^*) - \sum_j \lambda_j \nabla h_j(x^*) &= 0, \\ \mu_i g_i(x^*) &= 0, \quad \forall i \in I, \\ \mu_i &\geq 0, \quad \forall i \in I. \end{aligned}$$

These are the Kuhn-Tucker conditions.

**Definition 2.31.** [16] A vector field is a function that assigns a vector to each point in its domain. A vector field space the form

$$\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}.$$

---

We say that

(i) A vector field  $\vec{F}$  is a continuous if the component functions  $M, N, P$  are continuous.

(ii)  $\vec{F}$  is differentiable if the component functions  $M, N, P$  are differentiable.

**Definition 2.32.** [16] A function is of class  $C^n$  if it is differentiable  $n$  times and the  $n^{\text{th}}$  derivative is continuous.

## 2.1 Brownian motion and Markov process

In this section, we will discuss the meaning of Standard One-Dimensional BM and  $n$ -Dimensional BM. Brownian Motion was first observed in 1827 by the biologist Robert Brown. The definition and proving the existence of Brownian Motion was studied by Norbert Wiener in 1920. Hence, Brownian Motion is same as Wiener Process.

**Definition 2.33.** [12] A stochastic process

$$B = \{B_t : 0 \leq t \leq \infty\},$$

defined on probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  adapted to filtration  $F_t$  is named standard one-dimensional BM (Wiener process) if:

1.  $B$  are continuous function of  $t$ .
2.  $B_0 = 0$  a.s.
3. For  $0 \leq r \leq t$ ,  $B_t - B_r$  is independent of  $F_r$ .
4. For  $0 \leq r \leq t$ , the increment  $B_t - B_r$  is normally distribution with mean zero and variance  $t - r$ .

**Definition 2.34.** [8] The real-valued process  $(W_1(t), \dots, W_n(t))_{t \geq 0}$  is called an  $n$ -dimensional BM if

1.  $\forall k \in \{1, \dots, n\}$ ,  $(W_k(t))_{t \geq 0}$  is a Wiener process.
2.  $(W_k(t))_{t \geq 0}$ ,  $k = \bar{1}, \dots, n$ , are independent.

---

**Proposition 2.1.** [8]

1. *The Wiener process is a Gaussian process.*
2. *The Wiener process is a martingale.*
3. *Almost all paths of the BM  $(W_t)_{t \in \mathbb{R}_+}$  are not differentiable anywhere.*

**Definition 2.35.** [12]  $\mathcal{B}(\mathbb{R})$  is the usual Borel sigma algebra generated by the open sets.

**Definition 2.36.** [12]

$$X = \{X_t : 0 \leq t \leq \infty\},$$

defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  adapted to filtration  $F_t$  and take values in  $\mathbb{R}^d$  is called Markov process with initial distribution  $\mu$  if the following proprieties hold:

1.  $\mathbb{P}(X_0 \in A) = \mu(A)$  for any  $A \in \mathcal{B}(\mathbb{R}^d)$ .
2. if  $s, t > 0$  and  $A \in \mathcal{B}(\mathbb{R}^d)$ , Then

$$\mathbb{P}(X_{s+t} \in A | F_s) = \mathbb{P}(X_{s+t} \in A | X_s) \quad a.s.$$

The second property is called the Markov property (i.e. the conditional probability distribution of future states rely just on the present state).

## 2.2 Differential equations

In this section, we will discuss the difference between ordinary differential equation (ODE) and stochastic differential equation (SDE).

**Definition 2.37.** [1] *Differential equation is an equation that contains one or more terms and the derivatives of one variable.*



---

### 2.2.1 Ordinary differential equations

**Definition 2.38.** [1] ODE consist of only one independent variable and one or more of its derivatives relative to the variable (only ordinary derivatives appear in the equation).

Confirm a point  $x_0 \in \mathbb{R}^n$ . The ODE :

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{b}(\mathbf{x}(t)) & (t > 0) \\ \mathbf{x}(0) = x_0, \end{cases}$$

where

$$\mathbf{b} : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

is a smooth vector field. Also,

$$\mathbf{x}(\cdot) : [0, \infty) \rightarrow \mathbb{R}^n,$$

is the trajectory of the solution.

Below is a graph of a possible trajectory of the solution of ODE.

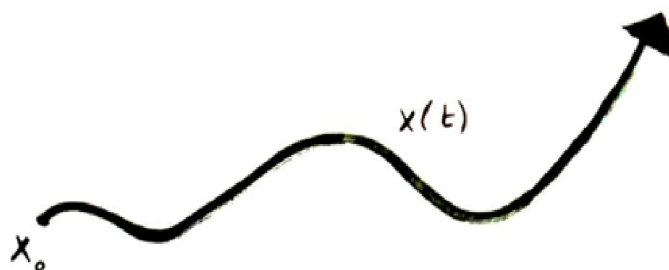


FIGURE 2.1: A SAMPLE PATH FOR A SOLUTION OF A DIFFERENTIAL EQUATION

**Definition 2.39.** [31] A 1<sup>st</sup> order linear ODE is linear has the form

$$\frac{dy}{dx} + A(x)y = B(x), \quad y(x_0) = y_0,$$

---

where  $y$  is the dependent variable and  $x$  is the independent variable.

Let us now solve the following ODE, which will appear in Chapter 2.

**Example 2.4.** *The following ODE*

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad S_0(0) = s_0 > 0. \quad (2.2)$$

Can be solved using the integrating factor method. First let us write it as the following

$$\frac{dS_0(t)}{dt} - r(t)S_0(t) = 0.$$

The integrating factor  $\mu(t)$  is computed as

$$\mu(t) = e^{-\int_0^t r(t)dt}.$$

The solution has the form

$$S_0(t) = \frac{1}{\mu(t)} \left( \int \mu(t) \cdot 0 dx + C \right),$$

where  $C$  is a constant

Hence,

$$S_0(t) = C e^{\int_0^t r(t)dt}.$$

Using that  $S_0(0) = s_0 > 0$  we obtain

$$S_0(t) = s_0 = C e^{\int_0^0 r(t)dt}.$$

Hence

$$C = s_0.$$

Finally,

$$S_0(t) = s_0 e^{\int_0^t r(t)dt}.$$

---

## 2.2.2 Stochastic differential equations

In this subsection we define stochastic differential equation and give an example illustrate it.

**Definition 2.40.** [12] *Any SDE is a differential equation with some coefficients are being random, and so its solution will have some randomness.*

*Stochastic differential equation is a modification on the ODE include random effect disturbing the system*

$$\begin{cases} \dot{\mathbf{X}}(t) = \mathbf{b}(\mathbf{X}(t)) + \mathbf{B}(\mathbf{X}(t))\boldsymbol{\xi}(t) & (t > 0) \\ \mathbf{X}(0) = x_0, \end{cases}$$

where

$$\mathbf{B} : \mathbb{R}^n \rightarrow \mathbb{M}^{n \times m}$$

and

$$\boldsymbol{\xi}(\cdot) := \text{“white noise”}.$$

Below is a graph of trajectory of the stochastic differential equation.



FIGURE 2.2: A SAMPLE PATH FOR A SOLUTION OF A STOCHASTIC DIFFERENTIAL EQUATION

**Example 2.5.** *Given the following differential equation*

$$\frac{dN}{dt} = a(t)N(t), \quad N(0) = N_0 \text{ (constant),}$$

---

where

1.  $N(t)$  is the size of the population at  $t$ .

2.  $a(t)$  is the relative rate of growth at  $t$  and

$$a(t) = r(t) + \text{“noise”}.$$

This model is named the population growth model so the previous DE becomes stochastic.

## 2.3 Itô's formula

In this section we will state Itô's formula which help us to find the derivatives of the SDE.

**Definition 2.41.** [12]  $\mathbb{L}^1(0, T)$  is the space of all real-valued, progressively measurable stochastic process  $F(\cdot)$  such that

$$E \left[ \int_X |F| dt \right] < \infty.$$

**Definition 2.42.** [12]  $\mathbb{L}^2(0, T)$  is the space of all real-valued, square-integrable functions defined on  $(0, T)$ .

**Definition 2.43.** [12] Suppose that  $X(\cdot)$  is a real-valued stochastic process satisfying

$$X(r) = X(s) + \int_s^r F dt + \int_s^r G dW \tag{2.3}$$

for some  $F \in \mathbb{L}^1(0, T)$ ,  $G \in \mathbb{L}^2(0, T)$  and all times  $0 \leq s \leq r \leq T$ . We say that  $X(\cdot)$  has the stochastic differential

$$dX = F dt + G dW, \quad 0 \leq t \leq T.$$

---

**Theorem 2.3.** [12] *Given the SDE*

$$dX = Fdt + GdW,$$

for  $F \in \mathbb{L}^1(0, T)$ ,  $G \in \mathbb{L}^2(0, T)$ . Assume  $u(x, t)$  defined by

$$u : \mathbb{R} \times [0, T] \rightarrow \mathbb{R},$$

has continuous 1<sup>st</sup> and 2<sup>nd</sup> derivative with respect to  $x$  and continuous 1<sup>st</sup> derivative with respect to  $t$  exist and are continuous. Put

$$Y(t) := u(X(t), t).$$

Implies  $Y$  has the SD

$$\begin{aligned} dY &= \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dX + 0.5 \frac{\partial^2 u}{\partial x^2} G^2 dt. \\ &= \left( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} F + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} G^2 \right) dt + \frac{\partial u}{\partial x} G dW. \end{aligned} \quad (2.4)$$

We call (2.4) Itô's formula. If we want to write the Itô's formula for one standard BM we use chain rule as follow. Let  $f(x)$  and  $Y(t)$  are a differentiable functions, then

$$\begin{aligned} \frac{d}{dt} f(Y(t)) &= f'(Y(t)) Y'(t) \\ &= f'(Y(t)) dY(t). \end{aligned}$$

However, if  $Y(t)$  is not differentiable and has nonzero quadratic variation, then the formula is

$$df(Y(t)) = f'(Y(t)) dB(t) + 0.5 f''(Y(t)) dt.$$

The above equation is give use an equivalent form and idea how we get (2.4).

---

**Remark 2.2.** Note that the identity (2.4) means that for all  $0 \leq s \leq r \leq T$ ,

$$\begin{aligned} Y(r) - Y(s) &= u(X(r), r) - u(X(s), s) \\ &= \int_s^r \frac{\partial u}{\partial t}(X, t) + \frac{\partial u}{\partial x}(X, t)F + \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(X, t)G^2 dt \\ &\quad + \int_s^r \frac{\partial u}{\partial x}(X, t)GdW \quad a.s. \end{aligned}$$

Since if we set  $s = 0$  in 2.3 we get

$$X(r) = X(0) + \int_0^r F dt + \int_0^r G dW$$

Concluding that  $X(\cdot)$  has continuous sample paths almost surely. Thus for almost every path, the functions

$$t \rightarrow u_t(X(t), t),$$

$$t \rightarrow u_x(X(t), t),$$

$$t \rightarrow u_{xx}(X(t), t),$$

are continuous; and so integrals in the statement of this remark are defined.

## *Chapter 3*

---

### *Optimal strategies within a social market of $L$ welfare providers and one risky asset*

---

In this Chapter, we extend work done by Moath introduced in [20] by allowing the welfare policy to be a control variable added to the problem of the economic agent. We assume that the economic agent has entered the welfare markets to save her family through a contracts which starts at time  $t = 0$  up until time  $T$  in the future, where  $T$  is the retirement age for the economic agent. The new setting in this chapter is that the economic agent has  $L$  welfare providers and want to choose one only at each instant of time  $t \in [0, T]$ . And the economic agent pay an amount  $q_l(t)$  where  $l = 1, 2, \dots, L$  to the  $l^{th}$  social welfare which is called the welfare premium. The welfare premium  $q_l(t)$  is based on a welfare rate  $h_l(t)$  determined by the  $l^{th}$  social welfare provider.

---

### 3.1 Model setup

Our industrial markets consists of the financial market which is available to the economic agent, the life insurance market and social welfare market. In this section we describe their details separately. After that, we introduce wealth process.

#### 3.1.1 Financial market with one risky asset

- Consider a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .
- $W(t)$  is a standard one-dimensional BM.
- Let

$$\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]},$$

be the  $\mathbb{P}$ - augmentation of the filtration generated by the one-dimensional BM denoted by  $W(\cdot)$ ,

$$(\sigma(W(s), s \leq t), \quad \forall t \geq 0.$$

- T interpreted as the economic agent retirement time.

Assume FM comprised of a risk-free asset with time- t price symbolized by  $(S_0(t))_{0 \leq t \leq T}$  that evolves with time by the ODE

$$dS_0(t) = S_0(t)r(t)dt, \quad s_0(0) = s_0, \tag{3.1}$$

and one risky security  $s_1(t)$  evolves with time according to the linear SDE:

$$\frac{dS_1(t)}{S_1(t)} = \mu(t)dt + \sigma(t)dW(t), \quad s_1(0) = s_1. \tag{3.2}$$



---

where  $S_0(0) = s_0$  and  $S_1(0) = s_1$  are a positive constant.

The expected rate of return for the risky asset is defined by

$$\mu : [0, T] \rightarrow R,$$

and assumed to be a continuous function named the risky security expected rate of return.  $\sigma(t)$  is the risky-asset volatility and

$$\sigma : [0, T] \rightarrow R,$$

is a continuous mapping verifying  $\sigma^2(t) \geq j, \forall t \in [0, T]$ , for some positive constant  $j$ .

**Assumption 3.1.** [25] *The coefficients  $r(t), \mu(t)$  and  $\sigma(t)$  are deterministic continuous functions on  $[0, T]$  such that the interest rate  $r(t) > 0$ .*

### 3.1.2 Life insurance market

Assume that the economic agent is still live at  $t = 0$ . In addition, her lifetime is a RV given by  $\tau \geq 0$  and defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ .

**Assumption 3.2.** [25] *The RV  $\tau$  has a distribution function*

$$G^- : [0, \infty] \rightarrow [0, 1],$$

*with density probability*

$$g(t) : [0, \infty) \rightarrow (0, \infty),$$

so

$$G^- \triangleq P(\tau \leq t) = \int_0^t g(u) du.$$

The insurance market contains a group of  $I$  companies each company offers a different contract. The life insurance is obtainable continuously by paying an

---

premium rate

$$\phi_i(t) : [0, T] \rightarrow (0, \infty)$$

for each  $i = 1, 2, \dots, I$  to the company.

If the economic agent perish at time  $\tau < T$  and she is still covered by insurance, the company pays:

$$\Upsilon(\tau) = \frac{\phi_i(\tau)}{\zeta_i(\tau)}, \quad (3.3)$$

where

$$\zeta_i : [0, T] \rightarrow (0, 1),$$

is a continuous and deterministic function named the  $i^{\text{th}}$  insurance premium-payout ratio.

**Assumption 3.3.** [25] *Assume that the  $I$  insurance companies suggest pairwise distinct contracts, i.e,  $\zeta_{i_1}(t) \neq \zeta_{i_2}(t)$  for every  $i_1 \neq i_2$  and Lebesgue-almost-every  $t \in [0, T]$ .*

$\phi(t)$  is written as a vector by

$$\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \phi_I(t) \end{pmatrix} \in (R_0^+)^I,$$

where for each  $i \in 1, 2, \dots, I$ . If the economic agent did not sign the contract with the  $i^{\text{th}}$  insurance company then  $\phi_i$  equals zeros. Otherwise it take the value that the economic agent paid. At each instant of time  $t \in [0, T]$ , the economic agent is assumed to sign only one contract.

---

### 3.1.3 Social welfare market

We assume that  $L$  social welfare providers offer services in the social welfare market. The economic agent aims to choose one social welfare which make the best advantage of this offer to protect her family. The contract between the economic agent and the social welfare market finishes as the economic agent dies or reaches to pension age, which occurs first.

If the economic agent dies at time  $\tau$  while participating in the social welfare policy, the social welfare provider pays

$$\frac{q_l(\tau)}{h_l(\tau)}, \quad (3.4)$$

where the *welfare rate*

$$h_l(t) : [0, T] \rightarrow (0, \infty),$$

is continuous and deterministic positive function for all  $l = 1, 2, \dots, L$  which is determined by the social welfare provider, and the *welfare premium*

$$q_l(t) : [0, T] \rightarrow (0, \infty),$$

is non-negative deterministic function for each  $l = 1, 2, \dots, L$ .

The economic agent total legacy to her estate at  $\tau \leq T$  is then given by

$$\tilde{Y}(\tau) = \sum_{l=1}^L \frac{q_l(\tau)}{h_l(\tau)}. \quad (3.5)$$

**Assumption 3.4.** *We assume that the  $L$  social welfare provider suggest pairwise distinct contracts, i.e,  $h_{l_1}(t) \neq h_{l_2}(t) \forall l_1 \neq l_2$  and Lebesgue a.e at  $t \in [0, T]$ .*

---

We can write  $q(t)$  as a vector,

$$q(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ \cdot \\ \cdot \\ \cdot \\ q_l(t) \end{pmatrix} \in R^L.$$

The quantity  $q_l(t), \forall l \in 1, 2, \dots, L$ , represent the life-insurance rate paid to the  $l^{\text{th}}$  insurance company. If the economic agent did not sign the contract with the insurance company then  $q_l$  equal zeros. Otherwise it take the value that the economic agent paid. We assume the economic agent signs only one contract at each instant of time  $t \in [0, T]$ .

Now we can denote the total legacy for the economic agent in the case of an early death at time  $\tau \leq T$  as:

$$X(\tau) + \sum_{i=1}^I \frac{\phi_i(\tau)}{\zeta_i(\tau)} + \sum_{l=1}^L \frac{q_l(\tau)}{h_l(\tau)},$$

where  $X(t)$  represent the economic agent's wealth savings at time  $t \in [0, T]$ .

Now we want to define a new function, which is called the survivor function denoted by  $G^+$  (i.e the probability that the lifetime is equal or greater than to  $t$ )

$$G^+(t) = 1 - G^-(t) = \int_t^{\infty} g(u) du,$$

The function  $\xi(t)$ , which is called hazard function is the ratio of the probability density function  $g(x)$  to the survival function:

$$\xi(t) = \frac{g(t)}{G^+(t)},$$

---

Also previous equation can be rewritten as:

$$\xi(t) = \frac{g(t)}{1 - G^-(t)}.$$

From the definition of survivor function, note that  $-g(t)$  is the derivative of  $G^+$  so we get:

$$\xi(t) = \frac{-\frac{dG^+}{dt}}{G^+(t)},$$

$$\xi(t) = -\frac{d}{dt}(\ln G^+(t)).$$

Integrate from 0 to t and introduce  $G^+(0) = 1$ :

$$-\int_0^t \xi(u)du = \ln G^+(t).$$

Take exponential function of both sides, so that

$$G^+(t) = e^{(-\int_0^t \xi(u)du)}. \quad (3.6)$$

Because of the probability density function is related to the hazard function, we obtain:

$$g(t) = \xi(t)e^{(-\int_0^t \xi(u)du)},$$

Hence, they are a relation between hazard functions and density functions. From now, the hazard function is known and defined as

$$\xi(t) : [0, \infty] \rightarrow (0, \infty),$$

continuous, deterministic function verifies:

$$\int_0^\infty \xi(t)dt = \infty.$$

---

Let  $G^+(k, t)$  be the conditional probability for the economic agent whose still live at time  $k$  conditional stilling live at  $t \leq k$ , given:

$$G^+(k, t) = P(\{\tau > k\} | \{\tau > t\}). \quad (3.7)$$

Similarly, we can define  $G^-(k, t)$  to be the the conditional probability for the economic agent time of death to happen at time  $k$  conditional to still live at  $t \leq k$ , as:

$$G^-(k, t) = P(\{\tau \leq k\} | \{\tau > t\}).$$

Also  $g^-(k, t)$  represent the corresponding density function with  $G^-(k, t)$ , then we get:

$$g^-(k, t) = \frac{d}{dt} G^-(k, t). \quad (3.8)$$

### 3.1.4 Wealth process

Assume the economic agent is started with initial wealth  $x$ . She will get an income  $m(t)$  through the period  $[0, T \wedge \tau]$ .  $m(t)$  stops by her death or her pension. That depends on what come first.

**Assumption 3.5.** [25]

$$m(t) : [0, T] \rightarrow [0, \infty),$$

*is a deterministic Borel-measurable function verifying*

$$\int_0^T m(t) dt < \infty,$$

*which is called the integrability condition.*

**Assumption 3.6.** [25]

1. Assume the consumption process  $(\kappa(t))_{0 \leq t \leq T}$  is a  $(\mathcal{F}_t)_{0 \leq t \leq T}$  progressively measurable non negative process verifying for  $T > 0$

---

$\int_0^T \kappa(t) dt$  is finite a.s.

2. The  $i^{\text{th}}$  insurance premium-payout ratio for each  $i = 1, 2, \dots, I$  and the  $l^{\text{th}}$  welfare premium-payout for each  $l = 1, 2, \dots, L$  are a non-negative  $(\mathcal{F}_t)_{0 \leq t \leq T}$  predictable process.
3. Let  $\theta_n(t)$  denote the fraction of the economic agent wealth allocated to the asset  $S_n$  at time  $t$ . Then the economic agent portfolio is given by

$$\Theta = \begin{pmatrix} \theta_1 \\ \cdot \\ \cdot \\ \cdot \\ \theta_N \end{pmatrix} \in \mathbb{R}^N,$$

If we know  $\theta_1(t), \dots, \theta_n(t)$ , then

$$\theta_0(t) = 1 - \sum_{n=1}^N \theta_n(t),$$

since all  $\theta_n(t) \in [0, 1]$ , for all  $n=0, 1, \dots, N$  and  $t \in [0, T]$ .

Moreover, we assume it a  $(\mathcal{F}_t)_{0 \leq t \leq T}$  progressively measurable satisfying:

$$\int_0^T \|\Theta(t)\|^2 dt \text{ is finite a.s.}$$

Now we can introduce the wealth process  $X(t)$ ,  $t \in [0, T \wedge \tau]$ .

$$\begin{aligned} X(t) = x_0 + \int_0^t \left( m(s) - \kappa(s) - \sum_{i=1}^I \phi_i(s) - \sum_{l=1}^L q(s) \right) ds \\ + \sum_{n=0}^N \int_0^t \frac{X(s) \theta_n(s)}{S_n(s)} dS_n(s). \end{aligned}$$

Since in this Chapter we have one risky asset ( $N=1$ ) above equation can be rewritten as

$$\begin{aligned}
X(t) = & x_0 + \int_0^t \left( m(s) - \kappa(s) - \sum_{i=1}^I \phi_I(s) - \sum_{l=1}^L q(s) \right) ds \\
& + \int_0^t \frac{X(s) \theta_0(s)}{S_0(s)} dS_0(s) + \int_0^t \frac{X(s) \theta_1(s)}{S_1(s)} dS_1(s). \tag{3.9}
\end{aligned}$$

Differentiate (3.9) relative to  $t$  we get

$$\begin{aligned}
\frac{dX}{dt} = & 0 + \left( m(t) - \kappa(t) - \sum_{i=1}^I \phi_i(t) - \sum_{l=1}^L q(t) \right) \\
& + \frac{X(t) \theta_0(t)}{S_0(t)} \frac{dS_0(t)}{dt} + \frac{X(t) \theta_1(t)}{S_1(t)} \frac{dS_1(t)}{dt}.
\end{aligned}$$

Substitute (3.1) and (3.2) in above equation we obtain

$$\begin{aligned}
\frac{dX}{dt} = & \left( m(t) - \kappa(t) - \sum_{i=1}^I \phi_i(t) - \sum_{l=1}^L q(t) \right) \\
& + \frac{X(t) \theta_0(t)}{S_0(t)} S_0(t) r(t) + \frac{X(t) \theta_1(t)}{S_1(t)} S_1(t) \left( \mu(t) + \sigma(t) \frac{dW(t)}{dt} \right).
\end{aligned}$$

Multiply by  $dt$  we get

$$\begin{aligned}
dX(t) = & \left( m(t) - \kappa(t) - \sum_{i=1}^I \phi_i(t) - \sum_{l=1}^L q(t) + \right. \\
& \left. \left( \theta_0(t) r(t) + \theta_1(t) \mu(t) \right) X(t) \right) dt + \theta_1(t) \sigma(t) X(t) dW(t). \tag{3.10}
\end{aligned}$$

## 3.2 Stochastic optimal control problem

In this section we will define the expected utility for the economic agent.

- Denote  $\mathcal{C}(0, x)$  is the set of all admissible decision strategies

$$(\kappa(\cdot), \phi(\cdot), q(\cdot), \theta(\cdot)).$$



- 
- $L(t, \cdot)$  represent the utility function for the economic agent's family consumption level at time  $t \in [0, T]$ .
  - $R(\cdot)$  represent the utility function for the terminal wealth at pension.
  - $Y(t, \cdot)$  represent the utility f for the amount of the economic agent's legacy at  $t \in [0, T]$  is denoted by .

Recall that that total legacy as said in previous subsection. Then we define the expected utility by:

$$E_{0,x} \left[ \int_0^{T \wedge \tau} L(s, \kappa(s)) ds + (Y(\tau, \Upsilon(\tau)) + Y(\tau, \tilde{\Upsilon}(\tau)) 1_{[0,T]}(\tau) + R(X(T)) 1_{(T,\infty)}(\tau)) \right],$$

where  $1_A$  is the indicator function of set A.

**Assumption 3.7.** [25] *The utility functions*

$$L(t, (\cdot)) : [0, T] \times [0, \infty) \rightarrow [0, \infty),$$

$$Y(t, (\cdot)) : [0, T] \times [0, \infty) \rightarrow [0, \infty),$$

*are twice differentiable, strictly increasing and strictly concave functions on their second variable. And*

$$R : [0, \infty) \rightarrow [0, \infty),$$

*is a twice differentiable, strictly increasing and strictly concave function.*

In this section we will use traditional methods to get explicit formulas for the economic agent's optimal decision strategies. In detail, we refer to Yee (2006) for driving the Hamilton -Jacobi- Bellman equation after that we solve it for the optimal strategies on a fixed planning horizon.

---

Let  $(\kappa(t), \phi(t), \theta(t), q(t)) \in \mathcal{C}(t, x)$  be the decision strategies for the dynamics of the wealth process with boundary condition  $X(t) = x$ . For any  $(\kappa(t), \phi(t), \theta(t), q(t)) \in \mathcal{C}(t, x)$  we define the corresponding expected utility as:

$$J(t, x; v) = E_{t,x} \left[ \int_t^{T \wedge \tau} L(s, \kappa(s)) ds + (Y(\tau, \Upsilon(\tau)) + Y(\tau, \bar{\Upsilon}(\tau))) 1_{[0,T]}(\tau) \right. \\ \left. + R(X_{t,x}^v(T)) 1_{(T,\infty)}(\tau) | \tau > t | \mathcal{F}_t \right], \quad (3.11)$$

where  $X_{t,x}^v(S)$  is the solution of the stochastic differential equation (SDE) (3.10). Depends on that all previous assumptions are hold 3.1-3.7, then we can get the next lemma.

**Lemma 3.1.** [20] *If  $\tau$  is independent of  $\mathbb{F}$ , then*

$$J(t, x; v) = E_{t,x} \left[ \int_t^T \left( G^+(k, t) L(k, \kappa(s)) + g^-(k, t) (Y(k, \Upsilon(s)) \right. \right. \\ \left. \left. + Y(k, \bar{\Upsilon}(s))) dk + G^+(T, t) R(X(T)) | \mathcal{F}_t \right], \right.$$

where  $G^+(k, t)$  and  $g^-(k, t)$  as in (3.7) and (3.8).

*Proof.* See [20]. □

The previous Lemma allow us to transform our OCP to a fixed planing horizon which is equivalent. To proceed let  $\mathcal{C}(t, x)$  be the set of all admissible strategies such that

$$v = (\kappa(\cdot), \phi(\cdot), q(\cdot), \theta(\cdot)) \in \mathcal{C}(t, x).$$

Then let

$$V(t, x) = \sup_{v \in \mathcal{C}(t, x)} J(t, x; v).$$

**Remark 3.2.** *We can write the conditional density function as*

$$g^-(k, t) = e^{-\int_t^s \xi(v) dv} g^-(k, s),$$

---

and similarly the conditional probability

$$G^+(k, t) = e^{-\int_t^s \xi(v) dv} G^+(k, s),$$

for  $0 \leq t < s < T$ .

**Lemma 3.3.** [20] Assume that all Assumptions 3.1-3.7 hold. Then for  $0 \leq t < s < T$ ,  $V(t, x)$  verifies the recursive relation

$$V(t, x) = \sup_{v \in \mathcal{C}(t, x)} \mathbb{E} \left[ \exp \left( - \int_t^s \xi(k) dk \right) V(s, X_{t,x}^v(s)) \right. \\ \left. + \int_t^s \left( G^+(k, t) L(k, \kappa(k)) + g^-(k, t) \left( Y(k, \Upsilon_{t,x}^v(k)) + Y(k, \bar{\Upsilon}_{t,x}^v(k)) \right) \right) dk \mid \mathcal{F}_t \right].$$

*Proof.* See [20]. □

### 3.3 Hamiltonian theorem

Let us define new function which is called the Hamiltonian function given by

$$\mathcal{H}(t, x, v) = \left( m(t) - \kappa(t) - \sum_{i=1}^I \phi_i(t) - \sum_{l=1}^L q(t) \right. \\ \left. + \left( r(t) + \theta(\mu(t) - r(t)) \right) x \right) V_x(t, x) + \frac{x^2}{2} (\theta(t)\sigma(t))^2 V_{xx}(t, x) \quad (3.12) \\ + L(t, c) + \xi(t) \left( Y \left( t, x + \sum_{i=1}^I \frac{\phi_i(t)}{\zeta_i(t)} \right) + Y \left( t, \sum_{l=1}^L \frac{q(t)}{h(t)} \right) \right).$$

The proof of the next result follows closely to the technique introduced by Ye [36] and Yong and Zhou [39] by adding the corresponding updates that fit with our model.

**Theorem 3.4. (Hamilton-Jacobi-Bellman Equation)**

---

Suppose that  $V$  is of class  $C^{1,2}([0, T] \times \mathbb{R})$ . Hence  $V$  verifies the HJB equation

$$V_t(t, x) - \xi(t)V(t, x) + \sup_{(\kappa, \phi, q, \theta) \in \mathbb{R} \times (\mathbb{R}^+)^I \times \mathbb{R}^L \times \mathbb{R}} \mathcal{H}(t, x, \kappa, \phi, q, \theta) = 0 \quad (3.13)$$

$$V(T, x) = R(x),$$

where the Hamiltonian function  $\mathcal{H}$  as in (3.12). Also,

$$v^* = (\kappa^*(\cdot), \phi^*(\cdot), q^*(\cdot), \theta^*(\cdot)) \in \mathcal{C}(t, x),$$

which related wealth is  $X^*$  is optimal  $\Leftrightarrow$  for a.e.  $s \in [t, T]$  we get

$$V_t(s, X^*(s)) - \xi(s)V(s, X^*(s)) + \mathcal{H}(s, X^*(s), v^*) = 0. \quad (3.14)$$

*Proof.* To prove this theorem we use the definition of Itô's formula, by setting  $s = t + h$  in the DPP.

$$\begin{aligned} V(t+h, X(t+h)) &= V(t, x) + \int_t^{t+h} \left\{ V_t(s, X(s)) \right. \\ &\quad + V_x(s, X(s)) \left[ r(s)X(s) - \kappa(s) - \sum_{i=1}^I \phi_i(s) - \sum_{l=1}^L q_l(s) \right. \\ &\quad \left. \left. + m(s) + \theta(s)(\mu(s) - r(s))X(s) \right] \right. \\ &\quad \left. + \frac{1}{2} V_{xx}(s, X(s)) \theta^2(s) \sigma^2(s) X^2(s) \right\} ds \\ &\quad + \int_t^{t+h} V_x(s, X(s)) \theta(s) \sigma(s) X(s) dW(s). \end{aligned} \quad (3.15)$$

We know that the Taylor expansion of  $e^x$  is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Hence,

$$e^{(-\int_t^{t+h} \xi(v) dv)} = 1 - \xi(t)h + O(h^2),$$

where  $h$  is small positive, and  $O(h^2)$  is an error of order two. From above equation

and by dynamic programming principle we obtain:

$$\begin{aligned}
0 &= \sup_{(\kappa, \phi, q, \theta) \in \mathcal{C}(t, x)} \mathbb{E} \left[ (1 - \xi(h) + O(h^2))V(t+h, X(t+h)) - V(t, x) \right. \\
&+ \int_t^{t+h} \left( G^+(u, t) L(u, \kappa(u)) \right. \\
&+ \left. \left. g^-(u, t) \left( Y(u, \Upsilon(u)) + Y(u, \tilde{\Upsilon}(u)) \right) \right) du \mid \mathcal{F}_t \right].
\end{aligned}$$

Divide above equation by  $h$ , let  $h \rightarrow 0$  and from PDD Lemma 3.3 we get

$$\begin{aligned}
0 &= \sup_{(\kappa, \phi, q, \theta) \in \mathcal{C}(t, x)} \left[ V_t(t, x) - \xi(t)V(t, x) + \left( m(t) - \kappa(t) - \sum_{i=1}^I \phi_i(t) \right. \right. \\
&- \left. \sum_{l=1}^L q(t) + \left( r(t) + \theta(\mu(t) - r(t)) \right) x \right) V_x(t, x) + \frac{1}{2} \sigma^2(t) \theta^2 x^2 V_{xx}(t, x) \\
&+ \left. L(t, \kappa(t)) + \xi(t) \left( Y(t, \Upsilon(t)) + Y(t, \tilde{\Upsilon}(t)) \right) \right].
\end{aligned}$$

Substitute  $\Upsilon(t)$  and  $\tilde{\Upsilon}(t)$  in above equation. Note that  $V_t(t, x) - \xi(t)V(t, x)$  doesn't rely on  $(\kappa, \phi, \theta, q)$  then we get the result.

To prove the 2<sup>nd</sup> part of the theorem which is equation (3.14) we also applying Itô's formula on  $e^{(-\int_t^s \xi(v)dv)} V(s, X(s))$  we get the following

$$\begin{aligned}
V(t, x) &= e^{-\int_t^T \xi(v)dv} W(X(T)) - \int_t^T e^{-\int_t^u \xi(v)dv} \left\{ V_t(u, X(u)) \right. \\
&+ V_x(u, X(u)) \left[ r(u)X(u) - \kappa(u) - \sum_{i=1}^I \phi_i(u) - \sum_{l=1}^L q(u) + m(x) \right. \\
&- \left. \left. r(u) \right) X(u) + \theta(u)(\mu(u)) \right] \xi(u) V(u, X(u)) \\
&+ \left. \frac{1}{2} V_{xx}(u, X(u)) \theta^2(u) \sigma^2(u) X^2(u) \right\} du \\
&- \int_t^T e^{-\int_t^u \xi(v)dv} V_x(u, X(u)) \theta(u) \sigma(u) X(u) dW(u).
\end{aligned}$$

---

Taking expectation to the both side

$$\begin{aligned}
V(t, x) &= \mathbb{E} \left[ e^{-\int_t^T \xi(v)dv} W(X(T)) - \int_t^T e^{-\int_t^u \xi(v)dv} \left\{ V_t(u, X(u)) \right. \right. \\
&\quad + V_x(u, X(u)) [r(u)X(u) - \kappa(u) - \sum_{i=1}^I \phi_u(u) - \sum_{l=1}^L q(u) + m(x) \\
&\quad - r(u)X(u) + \theta(u)(\mu(u))] \xi(u) V(u, X(u)) \\
&\quad \left. \left. + \frac{1}{2} V_{xx}(u, X(u)) \theta^2(u) \sigma^2(u) X^2(u) \right\} du \right. \\
&\quad \left. - \int_t^T e^{-\int_t^u \xi(v)dv} V_x(u, X(u)) \theta(u) \sigma(u) X(u) dW(u) \right].
\end{aligned}$$

By linearity of expectation we obtain:

$$\begin{aligned}
V(t, x) &= \mathbb{E} \left[ e^{-\int_t^T \xi(v)dv} W(X(T)) \right] - \mathbb{E} \left[ \int_t^T e^{-\int_t^u \xi(v)dv} \left\{ V_t(u, X(u)) \right. \right. \\
&\quad - \xi(u) V(u, X(u)) + V_x(u, X(u)) [m(u) - \kappa(u) - \sum_{i=1}^U \phi_u(u) \\
&\quad + \sum_{l=1}^L q(u) + r(u)X(u) + \theta(u)(\mu(u) - r(u))] \\
&\quad \left. \left. + \frac{1}{2} V_{xx}(u, X(u)) \theta^2(u) \sigma^2(u) X^2(u) du \right\} \right].
\end{aligned}$$

Rearrange terms we get

$$\begin{aligned}
V(t, x) &= J(t, x, \kappa, \phi, q, \theta) - \mathbb{E} \left[ \int_t^T e^{-\int_t^u \xi(v)dv} \left\{ V_t(u, X(u)) \right. \right. \\
&\quad \left. \left. - \xi(u) V(u, X(u)) + \mathcal{H}(u, X(u), \kappa, \phi, q, \theta) \right\} du \right].
\end{aligned}$$

From above equation we have

$$\begin{aligned}
V(t, x) &= J(t, x, \kappa^*, \phi^*, q^*, \theta^*) - \mathbb{E} \left[ \int_t^T e^{-\int_t^u \xi(v)dv} \left\{ V_t(u, X^*(u)) \right. \right. \\
&\quad \left. \left. - \xi(u) V(u, X^*(u)) + \mathcal{H}(u, X^*(u), \kappa^*, \phi^*, q^*, \theta^*) \right\} du \right].
\end{aligned}$$

---

From

$$V(t, x) - J(t, x, \kappa^*, \phi^*, q^*, \theta^*) \geq 0.$$

We prove that first direction as follow

$$\begin{aligned} & - \mathbb{E} \left[ \int_t^T e^{-\int_t^u \xi(v)dv} \left\{ V_t(u, X^*(u)) \right. \right. \\ & \left. \left. - \xi(u)V(u, X^*(u)) + \mathcal{H}(u, X^*(u), \kappa^*, \phi^*, q^*, \theta^*) \right\} du \right] \geq 0. \end{aligned}$$

Hence,

$$V_t(s, X^*(s)) - \xi(s)V(s, X^*(s)) + \mathcal{H}(s, X^*(s), v^*) \leq 0. \quad (3.16)$$

Now we want to prove the other direction.

$$\begin{aligned} & J(t, x, \kappa, \phi, q, \theta) - \mathbb{E} \left[ \int_t^T e^{-\int_t^u \xi(v)dv} \left\{ V_t(u, X(u)) \right. \right. \\ & \left. \left. - \xi(u)V(u, X(u)) + \mathcal{H}(u, X(u), \kappa, \phi, q, \theta) \right\} du \right] \end{aligned}$$

Rearrange terms we get

$$\begin{aligned} & \geq J(t, x, \kappa, \phi, q, \theta) - \mathbb{E} \left[ \int_t^T e^{-\int_t^u \xi(v)dv} \left\{ V_t(u, X(u)) \right. \right. \\ & \left. \left. - \xi(u)V(u, X(u)) + \phi_{(\kappa, \phi, q, \theta) \in \mathcal{C}(t, x)} \mathcal{H}(u, X(u), \kappa, \phi, q, \theta) \right\} du \right] \\ & = J(t, x, \kappa, \phi, q, \theta). \end{aligned}$$

Yielding

$$\begin{aligned} V(t, x) & \geq J(t, x, \kappa^*, \phi^*, q^*, \theta^*) - \mathbb{E} \left[ \int_t^T e^{-\int_t^u \xi(v)dv} \left\{ V_t(u, X^*(u)) \right. \right. \\ & \left. \left. - \xi(u)V(u, X^*(u)) + \mathcal{H}(u, X^*(u), \kappa^*, \phi^*, q^*, \theta^*) \right\} du \right] \end{aligned}$$

Hence

$$V_t(s, X^*(s)) - \xi(s)V(s, X^*(s)) + \mathcal{H}(s, X^*(s), v^*) \geq 0. \quad (3.17)$$

(3.16) and (3.17) implies the result.  $\square$

### 3.4 Optimal strategies in terms of the value function

In this subsection we want to find the optimal strategies such as the optimal insurance premium, optimal portfolio, optimal consumption, and optimal welfare policy in terms of the value function  $V(t, x)$  for the economic agent.

Let  $L_x(t, \cdot)$  and  $Y_x(t, \cdot)$  be the derivatives of the utility functions  $L(t, \cdot)$  and  $Y(t, \cdot)$  respectively. So the derivatives are invertible.

Let us define new unique functions  $Z_1, Z_2$  as

$$Z_1(t, L_x(t, x)) = x \quad \text{and} \quad L_x(t, Z_1(t, x)) = x,$$

$$Z_2(t, Y_x(t, x)) = x \quad \text{and} \quad Y_x(t, Z_2(t, x)) = x,$$

where

$$Z_1(t, x) : [0, T] \times [0, \infty) \rightarrow [0, \infty),$$

and

$$Z_2(t, x) : [0, T] \times [0, \infty) \rightarrow [0, \infty),$$

$\forall t \in [0, T]$  and  $x \in \mathbb{R}_0^+$ .

Next theorem give us the formula of the optimal strategies that we are looking for. The proof of the next result follows closely to the technique introduced by Mousa et al. [25] by adding the corresponding updates that fit with our model.

**Theorem 3.5.** *Let Assumptions 3.1-3.7 satisfied and  $V \in C^{1,2}([0, T] \times \mathbb{R}, \mathbb{R})$ . Implies that  $\mathcal{H}$  has a unique maximum  $v^* = (\kappa^*(\cdot), \phi^*(\cdot), q^*(\cdot), \theta^*(\cdot)) \in \mathcal{C}(t, x)$ . In addition, the optimal strategies that we are looking for are*

$$\kappa^*(t, x) = Z_1(t, V_x(t, x)),$$

$$\theta^*(t, x) = \frac{\beta_0 V_x(t, x)}{x V_{xx}(t, x) \sigma^2(t)},$$



---


$$\phi^*(t, x) = \begin{cases} \max \left\{ 0, \left[ Z_2 \left( t, \frac{\zeta_i(t)V_x(t, x)}{\xi(t)} \right) - x \right] \zeta_i(t) \right\}, & \text{if } i = i^*(t) \\ 0 & \text{Otherwise,} \end{cases}$$

where

$$i^*(t) = \arg \min_{i \in \{1, 2, \dots, I\}} \{\zeta_i(t)\}, \quad (3.18)$$

$$q^*(t, x) = \begin{cases} \max \left\{ 0, \left[ Z_2 \left( t, \frac{h_l(t)V_x(t, x)}{\xi(t)} \right) h_l(t) \right] \right\}, & \text{if } l = l^*(t) \\ 0 & \text{Otherwise,} \end{cases}$$

where

$$l^*(t) = \arg \min_{l \in \{1, 2, \dots, L\}} \{h_l(t)\}. \quad (3.19)$$

*Proof.* We want to find  $(\kappa^*(\cdot), \phi^*(\cdot), q^*(\cdot), \theta^*(\cdot)) \in \mathcal{C}(t, x)$  where  $\mathcal{H}$  attains its maximum. We can separate  $\mathcal{H}$  for 4 independent conditions as follows:

$$\begin{aligned} & \sup_{(\kappa, \phi, q, \theta) \in \mathbb{R} \times (\mathbb{R}_0^+)^I \times \mathbb{R}^L \times [0, 1]} \mathcal{H}(t, x, \kappa, \phi, q, \theta) = \sup_{\kappa \in \mathbb{R}} \left\{ L(t, \kappa) - \kappa V_x(t, x) \right\} \\ & + \sup_{\phi \in (\mathbb{R}_0^+)^I} \left\{ \xi(t) Y \left( t, x + \sum_{i=1}^I \frac{\phi_i(t)}{\zeta_i(t)} \right) - V_x(t, x) \sum_{i=1}^I \phi_i(t) \right\} \quad (3.20) \\ & + \sup_{q \in \mathbb{R}^L} \left\{ \xi(t) Y \left( t, \sum_{l=1}^L \frac{q(t)}{h_l(t)} \right) - \sum_{l=1}^L q(t) V_x(t, x) \right\} + m(t) V_x(t, x) + r(t) x V_x(t, x) \\ & + \sup_{\theta \in [0, 1]} \left\{ \frac{x^2}{2} \left( \theta(t) \sigma(t) \right)^2 V_{xx}(t, x) + \theta(t) (\mu(t) - r(t)) x V_x(t, x) \right\}. \end{aligned}$$

Now we study each variable separately. Let us start by differentiate equation (3.20) with respect to (w.r.t)  $\kappa$  we obtain :

$$L_\kappa(t, \kappa^*) - V_x(t, x) = 0.$$

Or

$$L_\kappa(t, \kappa^*) = V_x(t, x)$$

---

From the definition of  $Z_1$  and its uniqueness we get:

$$Z_1(t, L_\kappa(t, \kappa^*)) = Z_1(t, V_x(t, x)).$$

Thus

$$\kappa^*(t, x) = Z_1(t, V_x(t, x)).$$

Similarly, if we differentiate equation (3.20) w.r.t  $\theta$  gives us

$$x^2 V_{xx}(t, x) \theta^* \sigma^2 + (\mu(t) - r(t)) x V_x(t, x) = 0.$$

Now by solving above equation for the control variable  $\theta^*$  we get

$$\theta^*(t, x) = -\frac{(r(t) - (\mu(t)) V_x(t, x))}{x V_{xx}(t, x) \sigma^2(t)}, \quad (3.21)$$

For simplify set  $\beta_0 = \mu(t) - r(t)$ . Hence

$$\theta^*(t, x) = -\frac{\beta_0 V_x(t, x)}{x V_{xx}(t, x) \sigma^2(t)}, \quad (3.22)$$

To find the optimal values of  $\phi$  and  $q$  we solve the constrained optimization problem related to the control variable  $\phi \in (\mathbb{R}_0^+)^I$  and  $q \in \mathbb{R}^L$ , respectively, by using the Kuhn-Tucker conditions.

To solve the constrained optimization problem to find  $q^*$ . In Particular, we use the Kuhn-Tucker conditions to search for a solution

$$(q_1(t, x), \dots, q_L(t, x), \xi_1(t, x), \dots, \xi_L(t, x)),$$

subject to next qualities and inequalities

$$\frac{\xi(t)}{h_l(t)} Y_x \left( t, \sum_{l=1}^L \frac{q_l(t)}{h_l(t)} \right) - V_x(t, x) = -\xi_l,$$

---

subject to:

$$\begin{aligned} q_l &\geq 0, \\ \xi_l &\geq 0, \quad l = 1, 2, \dots, L \\ q_l \xi_l &= 0. \end{aligned}$$

We have two cases:

- Consider the case when  $l_1 = l_2$ . If  $\xi_{l_1}(t, x) = \xi_{l_2}(t, x)$  for some  $(t, x) \in [0, T] \times \mathbb{R}$ , we deduce that  $h_{l_1}(t, x) = h_{l_2}(t, x)$  and this contradicts our assumption that  $h_l(t)$  and  $h_k(t)$  are different for any  $l \neq k \in 1, 2, \dots, L$ .
- Hence, we conclude that  $l_1 \neq l_2$  for any  $l_1, l_2 \in \{1, 2, \dots, L\}$  and every  $x \in \mathbb{R}$ .

We get that  $\forall x \in \mathbb{R}$  and Lebesgue almost every  $t \in [0, T]$ ,  $\exists$  at most one  $l \in \{1, 2, \dots, L\}$  such that  $\xi_l(t, x) = 0$ . Hence, for Lebesgue a.e.  $t \in [0, T]$ ,  $\exists$  at most one  $l \in \{1, 2, \dots, L\}$  such that  $q_l(t, x) \neq 0$ .

Observe that

$$\begin{aligned} Z_2 \left( t, Y_x \left( t, \sum_{l=1}^L \frac{q_l(t)}{h_l(t)} \right) \right) &= Z_2 \left( t, (V_x(t, x) - \xi_{l_1}) \frac{h_{l_1}(t)}{\xi(t)} \right) \\ &= Z_2 \left( t, (V_x(t, x) - \xi_{l_2}) \frac{h_{l_2}(t)}{\xi(t)} \right). \end{aligned}$$

So we can get that

$$h_{l_1}(t) (V_x(t, x) - \xi_{l_1}) = h_{l_2}(t) (V_x(t, x) - \xi_{l_2}).$$

It's clear that if  $\xi_{l_1}(t, x) > \xi_{l_2}(t, x)$  for  $(t, x) \in [0, T] \times \mathbb{R}$ , then  $h_{l_1}(t) > h_{l_2}(t)$ . In addition, if we have  $\xi_{l_1}(t, x) = 0$  for some  $t \in [0, T]$ , then  $h_{l_1}(t) < h_{l_2}(t)$  for every  $l_2 \in \{1, 2, \dots, L\}$ .

Hence Let  $l^*(t)$  be given as

$$l^*(t) = \arg \min_{l \in \{1, 2, \dots, L\}}$$

---

Either  $q_l(t, x) = 0, \forall l \in \{1, 2, \dots, L\}$  or else  $q_{l^*(t)}(t, x) > 0$  is a solution to

$$Y_x \left( t, \frac{q_{l^*}(t)}{h_{l^*}(t)} \right) = \frac{h_{l^*}(t)V_x(t, x)}{\xi(t)}. \quad (3.23)$$

By the definition of  $Z_2$  and its uniqueness we get that

$$Z_2 \left( t, Y_x \left( t, \frac{\phi_{a^*}(t)}{\zeta_{k^*}(t)} \right) \right) = Z_2 \left( t, \frac{h_{l^*}(t)V_x(t, x)}{\xi(t)} \right).$$

Follows that

$$\frac{q_{l^*}(t)}{h_{l^*}(t)} = Z_2 \left( t, \frac{h_{l^*}(t)V_x(t, x)}{\xi(t)} \right).$$

Consequently

$$q^*(t, x) = Z_2 \left( t, \frac{h(t)V_x(t, x)}{\xi(t)} \right) h(t).$$

Getting

$$q_l^*(t, x) = \begin{cases} \max \left\{ 0, \left[ Z_2 \left( t, \frac{h_l(t)V_x(t, x)}{\xi(t)} \right) \right] h_l(t) \right\}, & \text{if } l = l^*(t) \\ 0 & \text{Otherwise.} \end{cases}$$

Finally do for  $\phi^*$ , differentiate equation (3.20) w.r.t  $\phi$

$$\frac{\xi(t)}{\zeta_i(t)} Y_X \left( t, x + \sum_{i=1}^I \frac{p_i(t)}{\zeta_i(t)} \right) - V_x(t, x).$$

Using same method previously, we search for

$$(\phi_1(t, x), \dots, \phi_i(t, x), \beta_1(t, x), \dots, \beta_I(t, x)),$$

veifying

$$\frac{\xi(t)}{\zeta_i(t)} Y_x \left( t, x + \sum_{i=1}^I \frac{\phi_n(t)}{\zeta_i(t)} \right) - V_x(t, x) = -\beta_i,$$

---

Subject to

$$\begin{aligned}\phi_i &\geq 0, \\ \beta_i &\geq 0, \quad i = 1, 2, \dots, I \\ \phi_i \beta_i &= 0.\end{aligned}$$

We have to cases:

- Consider the case when  $i_1 = i_2$ . Then  $\beta_{i_1}(t, x) = \beta_{i_2}(t, x)$  for some  $(t, x) \in [0, T] \times \mathbb{R}$ , we deduce that  $\zeta_{i_1}(t, x) = \zeta_{i_2}(t, x)$ . This contradicts the fact that all insurance companies suggest different agreements.
- Therefore, we just has the case where for any  $i_1, i_2 \in \{1, 2, \dots, I\}$  we have  $i_1 \neq i_2$  and every  $x \in \mathbb{R}$ ,  $\beta_{i_1}(t, x) \neq \beta_{i_2}(t, x)$  for Lebesgue almost every  $t \in [0, T]$ .

Get that at most one  $i \in \{1, 2, \dots, I\}$  such that  $\beta_i(t, x) = 0$ . So for Lebesgue a.e.  $t \in [0, T]$ ,  $\exists$  at most one  $i \in \{1, 2, \dots, I\}$  such that  $\phi_i(t, x) \neq 0$ .

By using the uniqueness function  $Z_2$  we get

$$\zeta_{i_1}(t) (V_x(t, x) - \beta_{i_1}) = \zeta_{i_2}(t) (V_x(t, x) - \beta_{i_2}).$$

Clearly, from above equation if  $\beta_{i_1}(t, x) > \beta_{i_2}(t, x)$  for  $(t, x) \in [0, T] \times \mathbb{R}$ , then  $\zeta_{i_1}(t) > \zeta_{i_2}(t)$ . In addition, if  $\beta_{i_1}(t, x) = 0$  for some  $t \in [0, T]$ , then we observe that  $\zeta_{i_1}(t) < \zeta_{i_2}(t)$  for every  $i_2 \in \{1, 2, \dots, I\}$ .

Hence, Let us define  $i^*(t)$  by

$$i^*(t) = \arg \min_{i \in \{1, 2, \dots, I\}} \{\zeta_i(t)\}.$$

Either  $\phi_i(t, x) = 0, \forall i \in \{1, 2, \dots, I\}$  or  $\phi_{i^*(t)}(t, x) > 0$  is a solution to

$$Y_x \left( t, x + \frac{\phi_{i^*(t)}}{\zeta_{i^*(t)}} \right) = \frac{\zeta_{i^*(t)} V_x(t, x)}{\xi(t)}. \quad (3.24)$$

---

Follows

$$x + \frac{\phi_{n^*}(t)}{\zeta_{n^*}(t)} = Z_2 \left( t, \frac{\zeta_{n^*}(t)V_x(t, x)}{\xi(t)} \right).$$

Yielding

$$\phi_i^*(t, x) = \begin{cases} \max \left\{ 0, \left[ Z_2 \left( t, \frac{\zeta_i(t)V_x(t, x)}{\xi(t)} \right) - x \right] \zeta_i(t) \right\}, & \text{if } i = i^*(t) \\ 0 & \text{Otherwise.} \end{cases}$$

Now we compute the 2<sup>nd</sup> derivative w.r.t each variable

$$\mathcal{H}_{\kappa\kappa}(t, x, v^*) = L_{\kappa\kappa}(t, \kappa^*),$$

it is negative from [3.7](#)

$$\mathcal{H}_{\phi_{k_1}\phi_{k_2}}(t, x, v^*) = \frac{\xi(t)}{\zeta_{i_1}(t)\zeta_{i_2}(t)} Y_{ZZ} \left( t, x + \frac{\phi_{k^*}^*(t)}{\zeta_{k^*}(t)} \right),$$

Note that  $\zeta_{n_1}(t)\zeta_{n_2}(t) > 0$ ,  $\xi(t) > 0$ , and  $Y$  is strictly concave so  $\mathcal{H}_{\phi_{k_1}\phi_{k_2}}(t, x, v^*)$  is negative.

Similarly,

$$\mathcal{H}_{q_1q_2}(t, x, v^*) = \frac{\xi(t)}{h_{l_1}(t)h_{l_2}(t)} Y_{\bar{Z}\bar{Z}} \left( t, \frac{q_{l^*}^*(t)}{h_{l^*}(t)} \right) < 0,$$

Finally,

$$\mathcal{H}_{\theta\theta}(t, x, v^*) = V_{xx}(t, x) \sigma^2 x^2.$$

Notice  $V_{xx}(t, x) < 0$ . Because of If  $V_{xx}(t, x) > 0$ , then  $\mathcal{H}$  wouldn't be bounded. Hence, by the HJB equation,  $V_t(t, x)$  or  $V(t, x)$  would have to be infinity. This contradicting the smoothness assumption we put on  $V$ . Hence we guarantee that  $\mathcal{H}_{\theta\theta}$  is negative. Hence  $\mathcal{H}$  has a unique regular interior maximum.  $\square$

---

### 3.5 The family of discounted constant relative risk aversion utilities

Here we characterize the optimal strategies in the case of discounted CRRA utilities function. Thus, we assume the utility functions for the economic agent as

$$\begin{aligned} L(t, \kappa) &= e^{-\rho t} \frac{\kappa^\gamma}{\gamma}, \\ Y(t, \Upsilon) &= e^{-\rho t} \frac{\Upsilon^\gamma}{\gamma}, \\ Y(t, \tilde{\Upsilon}) &= e^{-\rho t} \frac{\tilde{\Upsilon}^\gamma}{\gamma}, \\ R(X) &= e^{-\rho T} \frac{X^\gamma}{\gamma}. \end{aligned} \tag{3.25}$$

where  $\gamma$  is the risk aversion parameter and  $\gamma < 1$ ,  $\gamma \neq 0$ .  $\rho$  is positive denoted the discount rate, where  $\Upsilon$  and  $\tilde{\Upsilon}$  as in (3.3) and (3.4).

In the next section we state the optimal strategies for the family of CRRA utilities.

### 3.6 Explicit solution

In this section we will introduce the optimal strategies in terms of the parameters by solving explicitly the partial differential equation. Next proposition is the key result of this Chapter.

The proof of the next result follows closely to the technique introduced by Mousa et al. [25] by adding the corresponding updates that fit with our model.

---

**Proposition 3.1.** *Suppose that Assumptions 3.1-3.7 hold. Then the optimal strategies using (3.25) are*

$$\begin{aligned}
\kappa^*(t, x) &= \frac{1}{G(t)}(x + A(t)), \\
\theta^*(t, x) &= \frac{\beta(t)(x + A(t))}{x(1 - \gamma)\sigma^2(t)}, \\
\phi_i^*(t, x) &= \begin{cases} \max \left\{ 0, \zeta_{i(t)}(t) \left( \Psi(t)(x + A(t)) - x \right) \right\} & , \text{ if } i = i^*(t) \\ 0 & , \text{ otherwise } , \end{cases} \\
q_j^*(t, x) &= \begin{cases} \max \left\{ 0, h_{j(t)}(t) \left( E(t)(x + A(t)) \right) \right\} & , \text{ if } j = j^*(t) \\ 0 & , \text{ otherwise } , \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
A(t) &= \int_t^T m(s) e^{-\int_t^s (r(v) + \zeta_{i^*(v)}(v)) dv} ds, \\
\Psi(t) &= \frac{1}{G(t)} \left( \frac{\xi(t)}{\zeta_{i^*(t)}(t)} \right)^{\frac{1}{1-\gamma}}, \\
E(t) &= \frac{1}{G(t)} \left( \frac{\xi(t)}{h_{j^*}^\gamma(t)} \right)^{\frac{1}{1-\gamma}}, \\
G(t) &= e^{-\int_t^T \Pi(v) dv} + \int_t^T \Lambda(t) e^{-\int_t^s \Pi(v) dv} ds, \\
\Pi(t) &= \frac{\xi(t) + \rho}{1 - \gamma} - \frac{\gamma}{1 - \gamma} (r(t) + \zeta_{i^*(t)}(t)) + \frac{\gamma}{(1 - \gamma)^2} \Gamma, \\
\Lambda(t) &= 1 + \left( \frac{\zeta_{i^*(t)}^\gamma(t)}{\xi(t)} \right)^{\frac{1}{\gamma-1}} + \left( \frac{h_{j^*}^\gamma(t)}{\xi(t)} \right)^{\frac{1}{\gamma-1}}, \\
\Gamma(t) &= \frac{-\beta^2(t)}{2\sigma^2}.
\end{aligned}$$



---

*Proof.* Assume the utility function as given in (3.25) From Theorem 3.4 we have the following condition

$$L_\kappa(t, \kappa) - V_x(t, x) = 0, \quad (3.26)$$

$$\begin{aligned} x^2 V_{xx}(t, x) \theta \sigma^2 + (\mu(t) - r(t)) x V_x(t, x) &= 0, \\ \frac{\xi(t)}{h_j^*(t)} Y_x \left( t, \frac{q_j^*(t)}{h_j^*(t)} \right) - V_x(t, x) &= 0, \\ \frac{\xi(t)}{\zeta_i^*(t)} Y_x \left( t, x + \frac{p_i^*(t)}{\zeta_i^*(t)} \right) - V_x(t, x) &= 0. \end{aligned} \quad (3.27)$$

Differentiate  $L$  with respect to  $\kappa$

$$L_\kappa(t, \kappa) = e^{-\rho t} (\kappa^*(t, x))^{\gamma-1}.$$

Then substitute above equation in (3.26) we get

$$V_x(t, x) = e^{-\rho t} (\kappa^*(t, x))^{\gamma-1}.$$

Rearrange the above equation for  $\kappa$  we have

$$\kappa^*(t, x) = \left( e^{\rho t} V_x(t, x) \right)^{\frac{-1}{(1-\gamma)}}. \quad (3.28)$$

Note that in the second condition  $\theta^*$  is like as in Theorem 3.5

$$\theta^*(t, x) = -\frac{\beta_0 V_x(t, x)}{x V_{xx}(t, x) \sigma^2(t)}. \quad (3.29)$$

We have now to find the values of  $\phi$  and  $q$ . Differentiate  $Y$  with respect to  $x$  and the substitute it in (3.27) we obtain

$$e^{-\rho t} \left( x + \frac{\phi_{i^*}(t)}{\zeta_{i^*}(t)} \right)^{\gamma-1} = \frac{\zeta_{i^*}(t) V_x(t, x)}{\xi(t)},$$

where

$$Y_x(t, \Upsilon) = e^{-\rho t} \Upsilon^{\gamma-1}.$$

Hence

$$\phi_i^*(t, x) = \begin{cases} \max \left\{ 0, \left( \left( \frac{\zeta_i(t) e^{\rho t} V_x(t, x)}{\xi(t)} \right)^{\frac{-1}{(1-\gamma)}} - x \right) \zeta_i(t) \right\}, & \text{if } i = i^*(t) \\ 0 & \text{Otherwise.} \end{cases} \quad (3.30)$$

Similarly for  $q$ , Using the final condition we get

$$e^{-\rho t} \left( \frac{q_{l^*}(t)}{h_{l^*}(t)} \right)^{\gamma-1} = \frac{h_{l^*}(t)}{V_x(t, x)} \xi(t).$$

Consequently

$$q_l^*(t, x) = \begin{cases} \max \left\{ 0, \left( \left( \frac{h_l(t) e^{\rho t} V_x(t, x)}{\xi(t)} \right)^{\frac{-1}{(1-\gamma)}} \right) h_{l(t)}(t) \right\}, & \text{if } l = l^*(t) \\ 0 & \text{Otherwise.} \end{cases} \quad (3.31)$$

Now we are going to substitute (3.28), (3.29), (3.30) and (3.31) in the HJB equation (3.13) to find its solution.

$$\begin{aligned} \sup_{\nu \in \mathcal{C}} \mathcal{H}(t, x, \kappa, \phi, q, \theta) &= \sup_{\kappa \in \mathbb{R}} \left\{ L(t, \kappa) - \kappa V_x(t, x) \right\} + m(t) V_x(t, x) \\ &+ r(t) x V_x(t, x) + \sup_{\phi \in (\mathbb{R}_0^+)^I} \left\{ \xi(t) Y \left( t, x + \sum_{i=1}^I \frac{\phi_i(t)}{\zeta_i(t)} \right) - V_x(t, x) \sum_{i=1}^I \phi_i(t) \right\} \\ &+ \sup_{q \in \mathbb{R}^L} \left\{ \xi(t) Y \left( t, \frac{q(t)}{h(t)} \right) - q(t) V_x(t, x) \right\} \\ &+ \sup_{\theta \in [0,1]} \left\{ \frac{x^2}{2} \left( \theta(t) \sigma(t) \right)^2 V_{xx}(t, x) + \theta(t) (\mu(t) - r(t)) x V_x(t, x) \right\}. \end{aligned}$$

Yielding

$$\sup_{\nu \in \mathcal{C}} \mathcal{H}(t, x, \kappa, \phi, q, \theta) = \frac{e^{-\rho t} \left( e^{\rho t} V_x(t, x) \right)^{\frac{\gamma}{\gamma-1}}}{\gamma} - \left( e^{\rho t} V_x(t, x) \right)^{\frac{1}{\gamma-1}} V_x(t, x)$$

$$\begin{aligned}
& + \frac{x^2}{2} \left( -\frac{\beta_0 V_x(t, x)}{x V_{xx}(t, x) \sigma^2(t)} \right)^2 \sigma^2(t) V_{xx}(t, x) - \frac{\beta_0 V_x(t, x)}{x V_{xx}(t, x) \sigma^2(t)} \beta_0(t) x V_x(t, x) \\
& + \frac{e^{-\rho t} \xi(t)}{\gamma} \left( \frac{\zeta_{i^*}(t) e^{\rho t} V_x(t, x)}{\xi(t)} \right)^{\frac{\gamma}{\gamma-1}} - V_x(t, x) \left( \left( \frac{\zeta_{i^*}(t) e^{\rho t} V_x(t, x)}{\xi(t)} \right)^{\frac{1}{\gamma-1}} - x \right) \zeta_{i^*}(t) \\
& + \xi(t) e^{-\rho t} \frac{\left( \left( \frac{h_{l^*}(t) e^{\rho t} V_x(t, x)}{\xi(t)} \right)^{\frac{1}{\gamma-1}} \right)^\gamma}{\gamma} - V_x(t, x) \left( \left( \frac{h_{l^*}(t) e^{\rho t} V_x(t, x)}{\xi(t)} \right)^{\frac{1}{\gamma-1}} \right) h_{l^*}(t).
\end{aligned}$$

Rearrange the above terms we get

$$\begin{aligned}
\sup_{\nu \in \mathcal{C}} \mathcal{H}(t, x, \nu) &= V_x(t, x) \left( m(t) + x(\zeta_{i^*}(t) + r(t)) \right) + \frac{-\beta_0^2(t) V_x^2(t, x)}{2\sigma^2(t) V_{xx}(t, x)} \\
& + e^{\frac{\rho t}{\gamma-1}} \left( \frac{1-\gamma}{\gamma} \right) \left( V_x(t, x) \right)^{\frac{\gamma}{\gamma-1}} \left[ 1 + \frac{\left( \zeta_{i^*}(t) \right)^{\frac{\gamma}{\gamma-1}} + \left( h_{l^*}(t) \right)^{\frac{\gamma}{\gamma-1}}}{\left( \xi(t) \right)^{\frac{1}{\gamma-1}}} \right]. \quad (3.32)
\end{aligned}$$

For simplicity let

$$\begin{aligned}
\Lambda(t) &= \left[ 1 + \frac{\left( \zeta_{i^*}(t) \right)^{\frac{\gamma}{\gamma-1}} + \left( h_{l^*}(t) \right)^{\frac{\gamma}{\gamma-1}}}{\left( \xi(t) \right)^{\frac{1}{\gamma-1}}} \right]. \\
\Gamma(t) &= \frac{-\beta_0^2(t)}{2\sigma^2(t)}.
\end{aligned}$$

Equation (3.32) can be rewritten as

$$\begin{aligned}
& V_t(t, x) - \xi(t) V(t, x) + e^{\frac{\rho t}{\gamma-1}} \left( \frac{1-\gamma}{\gamma} \right) \left( V_x(t, x) \right)^{\frac{\gamma}{\gamma-1}} \Lambda(t) \\
& + V_x(t, x) \left( m(t) + x(\zeta_{i^*}(t) + r(t)) \right) + \Gamma(t) \frac{V_x^2(t, x)}{V_{xx}(t, x)} = 0, \quad (3.33)
\end{aligned}$$

with the terminal condition

$$V(T, x) = R(x). \quad (3.34)$$

---

In order to solve equation (3.33) we do the following steps

- Consider the ansatz function as

$$V(t, x) = \frac{a(t)}{\gamma} \left( A(t) + x \right)^\gamma,$$

- Find the derivatives of  $V_t$ ,  $V_x$  and  $V_{xx}$

$$\begin{aligned} V_t(t, x) &= a(t) \left( A(t) + x \right)^{\gamma-1} \frac{dA(t)}{dt} + \frac{1}{\gamma} \left( x + A(t) \right)^\gamma \frac{da(t)}{dt}, \\ V_x(t, x) &= a(t) \left( A(t) + x \right)^{\gamma-1}, \\ V_{xx}(t, x) &= (\gamma - 1) a(t) \left( A(t) + x \right)^{\gamma-2}. \end{aligned} \tag{3.35}$$

- Substitute above partial derivative in equation (3.33) to solve it, we get

$$\begin{aligned} & a(t) \left( A(t) + x \right)^{\gamma-1} \frac{dA(t)}{dt} + \frac{1}{\gamma} \left( A(t) + x \right)^\gamma \frac{da(t)}{dt} \\ & + a^{\frac{\rho t}{\gamma-1}} \left( \frac{1-\gamma}{\gamma} \right) \left( a(t) \left( A(t) + x \right)^{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} L(t) \\ & + a(t) \left( A(t) + x \right)^{\gamma-1} \left( m(t) + x(\zeta_{i^*}(t) + r(t)) \right) \\ & + \Gamma(t) \frac{\left( a(t) \left( A(t) + x \right)^{\gamma-1} \right)^2}{(\gamma - 1) a(t) \left( A(t) + x \right)^{\gamma-2}} - \xi(t) \frac{a(t)}{\gamma} \left( A(t) + x \right)^\gamma = 0. \end{aligned}$$

- Multiply previous equation  $\left(x + A(t)\right)^{-\gamma}$  we obtain

$$\begin{aligned}
& \frac{a(t)}{x + A(t)} \frac{dA(t)}{dt} + \frac{da(t)}{dt} \frac{1}{\gamma} - \xi(t) \frac{a(t)}{\gamma} \\
& + e^{\frac{-\rho t}{1-\gamma}} \left( \frac{1-\gamma}{\gamma} \right) \left( a(t) \right)^{\frac{-\gamma}{1-\gamma}} L(t) \\
& + \frac{a(t)}{x + A(t)} \left( i(t) + x(\zeta_{k^*}(t) + r(t)) \right) \\
& + \Gamma(t) \frac{a(t)}{\gamma - 1} = 0.
\end{aligned}$$

- Add  $\frac{r(t)a(t)A(t)}{A(t)+x}$  and  $\frac{\zeta_{i^*}(t)a(t)A(t)}{A(t)+x}$  to both sides we get

$$\begin{aligned}
& \frac{1}{\gamma} \frac{da(t)}{dt} - \xi(t) \frac{a(t)}{\gamma} + \frac{a(t)}{A(t) + x} \frac{dA(t)}{dt} + \frac{m(t)a(t)}{A(t) + x} \\
& + \frac{xr(t)a(t)}{x + A(t)} + \frac{r(t)a(t)A(t)}{x + A(t)} + \frac{x\zeta_{i^*}(t)u(t)}{x + A(t)} \\
& + \frac{\zeta_{i^*}(t)u(t)A(t)}{x + A(t)} + e^{\frac{-\rho t}{1-\gamma}} \left( \frac{1-\gamma}{\gamma} \right) \left( u(t) \right)^{\frac{-\gamma}{1-\gamma}} \Lambda(t + \Gamma(t)) \frac{G(t)}{\gamma - 1} \\
& = \frac{r(t)a(t)A(t)}{x + A(t)} + \frac{\zeta_{i^*}(t)u(t)A(t)}{x + A(t)}.
\end{aligned}$$

- To solve previous equation we separated it in two independent boundary value problem one for  $a(t)$  and other for  $A(t)$

$$\begin{aligned}
& \frac{1}{\gamma} \frac{da(t)}{dt} - \xi(t) \frac{a(t)}{\gamma} + \Gamma(t) \frac{a(t)}{\gamma - 1} + e^{\frac{-\rho t}{1-\gamma}} \left( \frac{1-\gamma}{\gamma} \right) a(t)^{\frac{-\gamma}{1-\gamma}} \Lambda(t) \\
& + \zeta_{i^*}(t)a(t) + r(t)a(t) = 0.
\end{aligned} \tag{3.36}$$

Rearrange above equation we obtain

$$\begin{aligned}
& \left( \zeta_{i^*}(t) + r(t) + \frac{\Gamma(t)}{\gamma - 1} - \frac{\xi(t)}{\gamma} \right) a(t) + e^{\frac{-\rho t}{1-\gamma}} \left( \frac{1-\gamma}{\gamma} \right) \left( a(t) \right)^{\frac{-\gamma}{1-\gamma}} \Lambda(t) + \frac{1}{\gamma} \frac{da(t)}{dt} = 0, \\
& a(T) = e^{-\rho T}.
\end{aligned} \tag{3.37}$$

---

And for  $A(t)$

$$\frac{(t)}{A(t) + X} \frac{dA(t)}{dt} + \frac{m(t)a(t)}{x + A(t)} - \frac{r(t)a(t)A(t)}{x + A(t)} - \frac{\zeta_{i^*}(t)u(t)A(t)}{x + A(t)} = 0.$$

Multiply above equation we get

$$\begin{aligned} \frac{dA(t)}{dt} + A(t) \left( -r(t) - \zeta_{i^*}(t) \right) + m(t) &= 0. \\ A(T) &= 0. \end{aligned} \quad (3.38)$$

- Let us start solving equation (3.38). Rewrite previous equation as:

$$\frac{dA(t)}{dt} + A(t) \left( -r(t) - \zeta_{i^*}(t) \right) = -m(t)$$

The above equation is 1<sup>st</sup> ODE which can be solve by integrating factor and the solution is in the form:

$$A(t) = \frac{1}{\mu(t)} \left( \int_t^T \mu(s) (-m(s)) ds + C \right),$$

where

$$\mu(t) = e^{-\int_t^s [r(u) + \zeta(u)] du}.$$

Hence,

$$A(t) = e^{\int_t^s [r(u) + \zeta(u)] du} \left( \int_T^t e^{-\int_t^s [r(u) + \zeta(u)] du} (-m(s)) ds + C \right).$$

Hence,

$$A(t) = e^{\int_t^s [r(u) + \zeta_{i^*}(u)] du} \left( \int_T^t e^{\int_t^s [-r(u) - \zeta_{i^*}(u)] du} (-m(s)) ds + C \right).$$

Using boundary condition we get:

$$A(t) = \int_t^T m(s) e^{-\int_t^s (r(u) + \zeta_{i^*}(u)) du} ds.$$

- To solve equation (3.36) we assume that the solution is on the form:

$$\begin{aligned} a(t) &= e^{-\rho t}(G(t))^{1-\gamma}, \\ a(T) &= e^{-\rho T}. \end{aligned} \tag{3.39}$$

Differentiate  $a(t)$  with respect to time  $t$ , we get

$$\frac{da(t)}{dt} = e^{-\rho t}(1-\gamma)(G(t))^{-\gamma} \frac{dG(t)}{dt} + (G(t))^{1-\gamma} (-\rho e^{-\rho t}).$$

Substitute the derivative of  $a(t)$  in equation (3.36) we obtain:

$$\begin{aligned} 0 &= \frac{1}{\gamma} \left( e^{-\rho t}(1-\gamma)(G(t))^{-\gamma} \frac{dG(t)}{dt} \right. \\ &- \left. \rho e^{-\rho t}(G(t))^{1-\gamma} \right) + \left( \zeta_{k^*}(t) + r(t) + \frac{\Gamma(t)}{\gamma-1} - \frac{\xi(t)}{\gamma} \right) e^{-\rho t}(G(t))^{1-\gamma} \\ &+ e^{\frac{-\rho t}{1-\gamma}} \left( \frac{1-\gamma}{\gamma} \right) \left( e^{-\rho t}(G(t))^{1-\gamma} \right)^{\frac{-\gamma}{1-\gamma}} \Lambda(t). \end{aligned}$$

Multiply the previous equation by  $\frac{\gamma}{1-\gamma}$ , we get;

$$\begin{aligned} &e^{-\rho t}(G(t))^{-\gamma} \frac{dG(t)}{dt} + (G(t))^{1-\gamma} \left( \frac{-\rho}{1-\gamma} e^{-\rho t} \right) \\ &\frac{\gamma}{1-\gamma} \left( \zeta_{k^*}(t) + r(t) + \frac{\Gamma(t)}{\gamma-1} - \frac{\xi(t)}{\gamma} \right) e^{-\rho t}(G(t))^{1-\gamma} \\ &+ e^{-\rho t} G(t)^{-\gamma} \Lambda(t) = 0. \end{aligned}$$

Divide the above equation by  $e^{-\rho t}$  we obtain:

$$\begin{aligned} &G(t)^{-\gamma} \frac{dG(t)}{dt} + (G(t))^{1-\gamma} \frac{-\rho}{1-\gamma} + G(t)^{-\gamma} \Lambda(t) \\ &\frac{\gamma}{1-\gamma} \left( \zeta_{k^*}(t) + r(t) + \frac{\Gamma(t)}{\gamma-1} - \frac{\xi(t)}{\gamma} \right) e^{-\rho t}(G(t))^{1-\gamma} = 0. \end{aligned}$$

---

Divide by  $G(t)^{-\gamma}$ , we get:

$$\frac{dG(t)}{dt} + G(t)^{\frac{-\rho}{1-\gamma}} + \Lambda(t) \frac{\gamma}{1-\gamma} \left( \zeta_{k^*}(t) + r(t) + \frac{\Gamma(t)}{\gamma-1} - \frac{\xi(t)}{\gamma} \right) e^{-\rho t} G(t) = 0.$$

Rearrange above equation:

$$\frac{dG(t)}{dt} + \left( \frac{\xi(t) + \rho}{1-\gamma} - \frac{1}{2} \gamma \left( \frac{\mu(t) - r(t)}{(1-\gamma)\sigma(t)} \right)^2 - \frac{\gamma}{1-\gamma} (r(t) + \zeta_{i^*}(t)) \right) G(t) + \Lambda(t) = 0. \quad (3.40)$$

For simplify let:

$$\Pi(t) \triangleq \frac{\xi(t) + \rho}{1-\gamma} - \frac{1}{2} \gamma \left( \frac{\mu(t) - r(t)}{(1-\gamma)\sigma(t)} \right)^2 - \frac{\gamma}{1-\gamma} (r(t) + \zeta_{i^*}(t)), \quad (3.41)$$

Equation (3.40) can be rewritten as:

$$\begin{aligned} \frac{dG(t)}{dt} - \Pi(t)G(t) + \Lambda(t) &= 0. \\ G(T) &= 1. \end{aligned}$$

Above equation is linear 1<sup>st</sup> ODE and its solution given by:

$$G(t) = e^{\int_t^T \Pi(u) du} \left( \int_T^t -\Lambda(s) e^{\int_T^s -\Pi(u) du} ds + c \right).$$

From  $G(T) = 1$  we have:

$$G(t) = e^{-\int_t^T \Pi(u) du} + \int_t^T \Lambda(s) e^{-\int_t^s \Pi(u) du} e^{\int_T^s \Pi(u) du} ds.$$

Thus

$$G(t) = e^{-\int_t^T \Pi(u) du} + \int_t^T \Lambda(s) e^{-\int_t^s \Pi(u) du} ds. \quad (3.42)$$

Finally :

$$a(t) = e^{-\rho t} \left( e^{-\int_t^T \Pi(u) du} + \int_t^T \Lambda(t)(s) e^{-\int_t^s \Pi(u) du} ds \right)^{1-\gamma}. \quad (3.43)$$



- 
- Substitute the value of  $V_x$  from (3.35) in (3.28) we obtain:

$$\kappa^*(t, x) = \frac{1}{G(t)}(A(t) + x).$$

- Substitute the value of  $V_x$  and  $V_{xx}$  from (3.35) in (3.29) we obtain:

$$\theta^*(t, x) = \frac{\beta(t)(A(t)) + x}{x(1 - \gamma)\sigma^2(t)}.$$

- For  $\phi$ , assume the case when  $i = i^*$ . After that substitute the value of  $V_x$  we get:

$$\phi_i^*(t, x) = \left( \frac{\left(\zeta_{u^*}(t)\right)^{\frac{1}{\gamma-1}}(G(t))^{-1}(x + A(t))}{\left(\xi(t)\right)^{\frac{1}{\gamma-1}}} - x \right) \zeta_{u^*}(t).$$

Simplify it by assume that:

$$\Psi(t) = \frac{1}{G(t)} \left( \frac{\zeta_{u^*}(t)}{\xi(t)} \right)^{\frac{1}{\gamma-1}}.$$

So we have:

$$\phi_i^*(t, x) = \begin{cases} \max \left\{ 0, \zeta_{i(t)}(t) \left( \Psi(t) (x + A(t)) - x \right) \right\} & , \text{if } i = i^*(t) \\ 0 & , \text{otherwise} . \end{cases}$$

- Similarly, we can find  $q$  by assuming the case where  $j = j^*$  and substitute the value of  $V_x$  we get:

$$q^*(t, x) = \frac{\left(h_{j^*}(t)\right)^{\frac{1}{\gamma-1}}(G(t))^{-1}(x + A(t))}{\left(\xi(t)\right)^{\frac{1}{\gamma-1}}} h_j(t).$$

Simplify it by assume that:

$$E(t) = \frac{1}{G(t)} \left( \frac{\xi(t)}{h_{j^*}^\gamma(t)} \right)^{\frac{1}{1-\gamma}}.$$

---

Thus

$$q_j^*(t, x) = \begin{cases} \max \left\{ 0, h_{j(t)}(t) \left( E(t) (x + A(t)) \right) \right\} & , \text{if } j = j^*(t) \\ 0 & , \text{otherwise .} \end{cases}$$

Hence, we conclude the proof.

□

## *Chapter 4*

---

*Generalized form: optimal strategies  
within a social market of  $L$  welfare  
providers and financial market of  $N$   
risky assets*

---

### **4.1 General framework**

Now, we are looking for the explicit solution in the case where the economic agent has access to  $L$  welfare providers and investing her saving in a financial market consists of  $N$  risky assets instead of only one security asset. To proceed we start by generalizing the industrial market models.

---

### 4.1.1 Generalized industrial market models

In this section , we introduce the financial market comprised of N risky assets. Also we state the corresponding wealth process.

#### 4.1.1.1 Financial market with N risky assets

Following the introduction represented in Chapter 3, Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space supplied with a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$  generated by M- dimensional BM  $W(\cdot), \sigma(W(s), s \leq t)$  for  $t \geq 0$ .

Consider the FM comprised of a risk-free asset and N risky securities, evolves according to the DE:

$$dS_0(t) = r(t)S_0(t)dt, \quad (4.1)$$

$$\frac{dS_n(t)}{S_n(t)} = \mu_n(t)dt + \sum_{m=1}^M \sigma_{nm}(t)dW_m(t), \quad (4.2)$$

where  $S_0(0) = s_0$  is a positive constant,  $r(t)$  is positive interest rate,

$$W(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \\ \cdot \\ \cdot \\ \cdot \\ W_M(t) \end{pmatrix} \in \mathbb{R}^M, \quad (4.3)$$

---

is  $M$ - dimensional BM,

$$\mu(t) = \begin{pmatrix} \mu_1(t) \\ \mu_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \mu_N(t) \end{pmatrix} \in \mathbb{R}^N, \quad (4.4)$$

is the vector of the risky-assets appreciation rates, and

$$\sigma(t) = \sigma_{nm}(t)_{1 \leq n \leq N, 1 \leq m \leq M}$$

is the  $N \times M$  matrix of risky-assets volatilities where  $n$  and  $m$  are natural numbers.

**Assumption 4.1.** [25] Assume that Assumption 3.1 hold. Besides to that

- The matrix  $(\sigma(t))^T \sigma(t)$  is non-singular matrix for Lebesgue almost all  $t \in [0, T]$  verifies

$$\sum_{m=1}^M \sum_{n=1}^N \int_0^T \sigma_{nm}^2(t) dt < \infty.$$

- $\exists (\mathcal{F}_t)_{0 \leq t \leq T}$ -progressively measurable  $\pi(t) \in \mathbb{R}^M$ , named the market price of risk, where the risk premium

$$\beta(t) = \begin{pmatrix} \mu_1(t) - r(t) \\ \mu_2(t) - r(t) \\ \cdot \\ \cdot \\ \cdot \\ \mu_N(t) - r(t) \end{pmatrix} \in \mathbb{R}^N, \quad (4.5)$$

is attached to  $\pi(t)$  by

$$\beta(t) = \pi(t)\sigma(t) \quad a.s.$$

Moreover, we assume

$$\int_0^T \pi^2(t) dt < \infty \quad a.s.,$$

---

and the following exponential martingale condition holds

$$\mathbb{E}\left[e^{-\int_0^T \pi(t)dW(t) - \frac{1}{2} \int_0^T \pi^2(t)dt}\right] = 1 .$$

We assume that all assumptions and concepts in this Chapter are the same as in Chapter 3.

#### 4.1.2 Generalized wealth process

In this subsection, we state the wealth process for the economic agent who start with initial value  $x_0$  and take an income  $m(t)$ , where  $t \in [0, \min\{\tau, T\}]$  .

Assume that Assumptions 3.5, 3.6 holds. Now, we can define the wealth process for the economic agent for  $0 \leq t \leq \min\{\tau, T\}$  as

$$\begin{aligned} X(t) = & x + \int_0^t \left( m(s) - \kappa(s) - \sum_{i=1}^I \phi_i(s) - \sum_{l=1}^L q_l(s) \right) ds \\ & + \sum_{n=0}^N \int_0^t \frac{\theta_n(s)X(s)}{S_n(s)} dS_n(s). \end{aligned}$$

Above equation can be rewrite as

$$\begin{aligned} X(t) = & x + \int_0^t \left( m(s) - \kappa(s) - \sum_{i=1}^I \phi_i(s) - \sum_{l=1}^L q_l(s) \right) ds \\ & + \int_0^t \frac{\theta_0(s)X(s)}{S_0(s)} dS_0(s) + \sum_{n=1}^N \int_0^t \frac{\theta_n(s)X(s)}{S_n(s)} dS_n(s). \end{aligned}$$

Substitute (4.1) and(4.2) in above equation we get

$$\begin{aligned} X(t) = & x + \int_0^t \left( m(s) - \kappa(s) - \sum_{i=1}^I \phi_i(s) - \sum_{l=1}^L q_l(s) \right) ds \\ & + \int_0^t \frac{\theta_0(s)X(s)}{S_0(s)} r(s) S_0(s) ds \\ & + \sum_{n=1}^N \int_0^t \frac{\theta_n(s)X(s)}{S_n(s)} S_n(s) \left( \mu_n(s) ds + \sum_{m=1}^M \sigma_{nm}(t) dW_m(s) \right). \end{aligned}$$

---

Rearrange above equation we have

$$\begin{aligned}
X(t) = & x + \int_0^t \left( m(s) - \kappa(s) - \sum_{i=1}^I \phi_i(s) - \sum_{l=1}^L q_l(s) \right) ds \\
& + \int_0^t \theta_0(s) X(s) r(s) ds \\
& + \sum_{n=1}^N \int_0^t \theta_n(s) X(s) \left( \mu_n(s) ds + \sum_{m=1}^M \sigma_{nm}(s) dW_m(s) \right).
\end{aligned}$$

Differentiate above equation relative to  $t$  we obtain

$$\begin{aligned}
dX(t) = & \left( m(t) - \kappa(t) - \sum_{i=1}^I \phi_i(t) - \sum_{l=1}^L q_l(t) + (\theta_0(t)r(t) \right. \\
& \left. + \sum_{n=1}^N \theta_n(t) \mu_n(t) \right) X(t) dt \\
& + \sum_{n=1}^N \theta_n(t) X(t) \sum_{m=1}^M \sigma_{nm}(t) dW_m(t).
\end{aligned} \tag{4.6}$$

## 4.2 Generalized stochastic optimal control problem

In this section, we will illustrate the optimal control problem (OCP) for the economic agent whose aim is to get the optimal strategies which gives her best advantage.

Denote  $\mathcal{C}(0, x)$  the set of all admissible decision strategies  $(\kappa(\cdot), \phi(\cdot), q(\cdot), \theta(\cdot))$ ,  $L(t, \cdot)$  represent the utility function for the economic agent's family consumption level at time  $t \in [0, T]$ ,  $R(\cdot)$  represent the utility function for the terminal wealth at pension time  $T$ , and the utility function for the size of the economic agent's legacy at time  $t \in [0, T]$  is denoted by  $Y(t, \cdot)$ . Recall that that total legacy as said in previous subsection. The expected utility defined by:

$$E_{0,x} \left[ \int_0^{T \wedge \tau} L(k, c(k)) dk + (Y(\tau, \Upsilon(\tau)) + Y(\tau, \tilde{\Upsilon}(\tau))) I_{[0,T]}(\tau) \right]$$

---


$$\left. + R(X(T)) I_{(T,\infty)}(\tau) \right].$$

As Chapter 3, we resort to the DPP in order to transform our complicated problem to an easier problem whose solution is a recursive relation.

We can generalize the previous expected utility in the last section as

$$J(t, x; v) = E_{t,x} \left[ \int_t^{T \wedge \tau} L(k, \kappa(k)) dk + (Y(\tau, \Upsilon(\tau)) + Y(\tau, \tilde{\Upsilon}(\tau))) I_{[0,T]}(\tau) \right. \\ \left. + R(X_{t,x}^v(T)) I_{(T,\infty)}(\tau) \mid \tau > t, \mathcal{F}_t \right], \quad (4.7)$$

where  $X_{t,x}^v(S)$  is the solution of the stochastic differential equation (SDE) (4.6), and  $\mathcal{C}(t, x)$  is the set of admissible decision strategies starting at time  $t$ .

### 4.3 Dynamic programming principle

In this section we will state the DPP Lemma and derive the corresponding HJB equation which help us to get the solution and the value function. Define

$$V(t, x) = \sup_{v \in \mathcal{C}(t,x)} J(t, x; v),$$

where

$$v = (\kappa(\cdot), \phi(\cdot), q(\cdot), \theta(\cdot)) \in \mathcal{C}(t, x).$$

Using Lemma 3.1 and DPP 3.3 we use the same technique as in previous section to derive the corresponding HJB equation, a 2<sup>nd</sup> order nonlinear PDE whose solution is the desired objective function, for the general case. The proof of the following general theorem follows similarly as the proof of Theorem 3.4.



---

**Theorem 4.1.** (*Shrateh-Mousa Th1*). Suppose that  $V$  is of class  $C^{1,2}([0, T] \times \mathbb{R}, \mathbb{R})$ . Hence  $V$  verifies the HJB equation

$$\begin{cases} V_t(t, x) - \xi(t)V(t, x) + \sup_{(\kappa, \theta, \phi, q) \in \mathbb{R} \times \mathbb{R}^N \times (\mathbb{R}_0^+)^I \times \mathbb{R}^L} H(t, x; \kappa, \theta, \phi, q) = 0 \\ V(T, x) = R(x), \end{cases}$$

where the Hamiltonian function  $\mathcal{H}$  is given by

$$\begin{aligned} \mathcal{H}(t, x; v) = & \left( m(t) - \kappa(t) - \sum_{i=1}^I \phi_i - \sum_{l=1}^L q_l \right. \\ & \left. + \left( r(t) + \sum_{n=1}^N \theta_n (\mu_n(t) - r(t)) \right) x \right) V_x(t, x) \\ & + \frac{x^2}{2} \sum_{m=1}^M \left( \sum_{n=1}^N \theta_n \sigma_{nm}(t) \right)^2 V_{xx}(t, x) \\ & + L(t, \kappa) + \xi(t)Y \left( t, x + \sum_{i=1}^I \frac{\phi_i(t)}{\zeta_i(t)} \right) + Y \left( t, \sum_{l=1}^L \frac{q_l(t)}{h_l(t)} \right). \end{aligned}$$

In addition,

$$v^* = (\kappa^*(\cdot), \phi^*(\cdot), q^*(\cdot), \theta^*(\cdot)) \in \mathcal{C}(t, x),$$

is optimal if and only if for a.e.  $s \in [t, T]$  we have

$$V_s(s, X^*(s)) - \xi(s)V(s, X^*(s)) + H(s, X^*(s); v^*) = 0.$$

#### 4.4 Generalized optimal strategies in terms of the value function

In this subsection we want to find the optimal strategies such as the optimal insurance premium, optimal portfolio, optimal consumption, and optimal welfare policy for the economic agent.

Recall that we denote the utility function describing the economic agent's consumption by  $L(t, \cdot)$ , and the utility function describing the size of the economic agent's legacy by  $Y(t, \cdot)$  for all  $0 \leq t \leq T$ . Also, assume Assumption 3.7 hold. Let  $L_x(t, \cdot)$

---

and  $Y_x(t, \cdot)$  symbolised the derivatives of  $L(t, \cdot)$  and  $Y(t, \cdot)$  respectively. So the derivatives are invertible.

Let us define a unique function  $Z_1, Z_2$  as we define it in Chapter 3.

$$Z_1(t, L_x(t, x)) = x \quad \text{and} \quad L_x(t, Z_1(t, x)) = x, \quad (4.8)$$

$$Z_2(t, Y_x(t, x)) = x \quad \text{and} \quad Y_x(t, Z_2(t, x)) = x, \quad (4.9)$$

where

$$Z_1(t, x) : [0, T] \times [0, \infty) \rightarrow [0, \infty),$$

and

$$Z_2(t, x) : [0, T] \times [0, \infty) \rightarrow [0, \infty),$$

$\forall t \in [0, T]$  and  $x \in \mathbb{R}_0^+$ .

Next theorem give us the formula of the optimal strategies of the objective function and its derivatives. The proof of the next result follows closely to the technique introduced by Mousa et al. [25] by adding the corresponding updates that fit with our model.

**Theorem 4.2.** (Shratch-Mousa Th2). *Let  $V$  is of class  $C^{1,2}([0, T] \times \mathbb{R}, \mathbb{R})$ . As a result the Hamiltonian function  $H$  given in the statement of Theorem 4.1 has a unique maximum*

$$v^* = (\kappa^*(\cdot), \theta^*(\cdot), \phi^*(\cdot)) \in \mathcal{C}(t, x).$$

In addition, the optimal strategies are

$$\kappa^*(t, x) = Z_1(t, V_x(t, x)),$$

$$\theta^*(t, x) = -\frac{V_x(t, x)}{xV_{xx}(t, x)}\varepsilon\beta(t),$$

and, for each  $i \in \{1, 2, \dots, I\}$ , we have that

$$\phi_i^*(t, x) = \begin{cases} \max \{0, [Z_2(t, \zeta_i(t)(\xi(t))^{-1}V_x(t, x)) - x] \zeta_i(t)\}, & \text{if } i = i^*(t) \\ 0 & \text{otherwise.} \end{cases}$$

where

$$i^*(t) = \arg \min_{i \in \{1, 2, \dots, I\}} \{\zeta_i(t)\}$$

and, for each  $l \in \{1, 2, \dots, L\}$ , we have that

$$q^*(t, x) = \begin{cases} \max \left\{ 0, \left[ Z_2 \left( t, \frac{h_l(t)V_x(t, x)}{\xi(t)} \right) h_l(t) \right] \right\}, & \text{if } l = l^*(t) \\ 0 & \text{otherwise.} \end{cases}$$

where

$$l^*(t) = \arg \min_{l \in \{1, 2, \dots, L\}} \{h_l(t)\}.$$

where  $\varepsilon$  is the non-singular square matrix  $(\sigma\sigma^T)^{-1}$  and  $\beta(t)$  same as (4.5).

*Proof.* Note that  $\kappa^*(\cdot)$ ,  $\phi^*(\cdot)$ , and  $q^*(\cdot)$  are same as Theorem 3.5. So the different is when we looking for  $\theta^*(\cdot)$ .

Let us separate  $\mathcal{H}$  as follow

$$\begin{aligned} & \sup_{(\kappa, \theta, \phi, q) \in \mathbb{R}^{N+1} \times \mathbb{R}^N \times (\mathbb{R}_0^+)^K \times \mathbb{R}^L} H(t, x; v) \\ &= \sup_{\kappa \in \mathbb{R}} \{L(t, \kappa) - cV_x(t, x)\} + r(t)xV_x(t, x) \\ & \quad + \sup_{\phi \in (\mathbb{R}_0^+)^I} \left\{ \xi(t)Y \left( t, x + \sum_{i=1}^I \frac{\phi_i}{\zeta_i(t)} \right) - V_x(t, x) \sum_{i=1}^I \phi_i \right\} \\ & \quad + m(t)V_x(t, x) + \sup_{\theta \in \mathbb{R}^N} \left\{ \frac{x^2}{2} \sum_{m=1}^M \left( \sum_{n=1}^N \theta_n \sigma_{nm}(t) \right)^2 \right. \\ & \quad \left. \times V_{xx}(t, x) + \sum_{n=1}^N \theta_n (\mu_n(t) - r(t)) xV_x(t, x) \right\} \\ & \quad + \sup_{q \in \mathbb{R}^L} \left\{ \xi(t)Y \left( t, \sum_{l=1}^L \frac{q_l(t)}{h_l(t)} \right) - \sum_{l=1}^L q_l(t)V_x(t, x) \right\} \end{aligned} \tag{4.10}$$

We start with finding the optimal strategy for insurance premium payments and social welfare premium payments since its similarly to Chapter 3. we use the

---

Kuhn-Tucker conditions to search for a solution

$$(q_1(t, x), \dots, q_L(t, x), \lambda_1(t, x), \dots, \lambda_L(t, x)),$$

$$(\phi_1(t, x), \dots, \phi_I(t, x), \mu_1(t, x), \dots, \mu_K(t, x)).$$

To the following qualities and inequalities

$$\frac{\xi(t)}{h_l(t)} Y_X \left( t, \sum_{l=1}^L \frac{q_l(t)}{h_l(t)} \right) - V_x(t, x) = -\lambda_l,$$

subject to:

$$q_l \geq 0,$$

$$\lambda_l \geq 0, \quad l = 1, 2, \dots, L$$

$$q_l \lambda_l = 0.$$

And

$$\frac{\lambda(t)}{\zeta_i(t)} Y_Z \left( t, x + \sum_{i=1}^I \frac{\phi_i(t)}{\zeta_i(t)} \right) - V_x(t, x) = -\mu_i,$$

Subject to

$$\phi_i \geq 0,$$

$$\mu_i \geq 0, \quad i = 1, 2, \dots, I$$

$$\phi_i \mu_i = 0.$$

Do similarly as Chapter 3, we get the result in proposition.

Now, we want to find the optimal strategies for the consumption  $c^*(t)$  Computing the first-order conditions with respect to  $\kappa$  we obtain the following:

$$L_\kappa(t, \kappa^*) - V_x(t, x) = 0.$$

Or

$$L_\kappa(t, \kappa^*) = V_x(t, x)$$

---

From the definition of  $Z_1$  and its uniqueness we get:

$$\kappa_1(t, L_\kappa(t, \kappa^*)) = Z_1(t, V_x(t, x)).$$

Thus

$$\kappa^*(t, x) = Z_1(t, V_x(t, x)).$$

To find  $\theta^*$  Computing the first-order conditions with respect to  $\theta$  we get:

$$xV_x(t, x)\beta + x^2V_{xx}(t, x)\sigma\sigma^T\theta^* = 0_{\mathbb{R}^N}$$

OR

$$xV_x(t, x)\beta = -x^2V_{xx}(t, x)\sigma\sigma^T\theta^*$$

where  $\beta$  denotes the risk premium given in origin of  $\mathbb{R}^N$ .

Rearrange above equation we have

$$\theta^*(t, x) = -\frac{V_x(t, x)}{xV_{xx}(t, x)}\varepsilon\beta(t)$$

Now we compute the 2<sup>nd</sup> derivative w.r.t each variable

$$\mathcal{H}_{\kappa\kappa}(t, x, v^*) = L_{\kappa\kappa}(t, \kappa^*),$$

it is negative from [3.7](#)

$$\mathcal{H}_{\phi_{i_1}\phi_{i_2}}(t, x, v^*) = \frac{\xi(t)}{\zeta_{i_1}(t)\zeta_{i_2}(t)}Y_{ZZ}\left(t, x + \frac{\phi_{i^*}^*(t)}{\zeta_{i^*}(t)}\right),$$

Note that  $\zeta_{i_1}(t)\zeta_{i_2}(t) > 0$ ,  $\xi(t) > 0$ , and  $Y$  is strictly concave so  $\mathcal{H}_{\phi_{i_1}\phi_{i_2}}(t, x, v^*)$  is negative.

Similarly,

$$\mathcal{H}_{q_{i_1}q_{i_2}}(t, x, v^*) = \frac{\xi(t)}{h_{i_1}(t)h_{i_2}(t)}Y_{\bar{Z}\bar{Z}}\left(t, \frac{q_{i^*}^*(t)}{h_{i^*}(t)}\right) < 0,$$

Finally,

$$\mathcal{H}_{\theta\theta}(t, x, v^*) = V_{xx}(t, x)\sigma\sigma^Tx^2.$$

---

Notice  $V_{xx}(t, x) < 0$ . Because of If  $V_{xx}(t, x) > 0$ , then  $\mathcal{H}$  wouldn't be bounded. Hence, by the HJB equation,  $V_t(t, x)$  or  $V(t, x)$  would have to be infinity. This contradicting the smoothness assumption we put on  $V$ . Hence we guarantee that  $\mathcal{H}_{\theta\theta}$  is negative. Hence  $\mathcal{H}$  has a unique regular interior maximum.  $\square$

## 4.5 Explicit solution for the generalized optimal strategies

*In this section, we represent the optimal strategies in the case of discounted CRRA utilities function (3.25).*

*In the next proposition we state the optimal strategies for the family of CRRA utilities. The proof of the next result follows closely to the technique introduced by Mousa et al. [25] by adding the corresponding updates that fit with our model.*

**Proposition 4.1.** *(Shratch-Mousa Th3). Let  $\varepsilon$  is the non-singular square matrix  $(\sigma\sigma^T)^{-1}$  The optimal strategies in the case of discounted CRRA utilities function are*

$$\begin{aligned}\kappa^*(t, x) &= \frac{1}{G(t)}(x + A(t)), \\ \theta^*(t, x) &= \frac{1}{1 - \gamma} \frac{x + A(t)}{x} \varepsilon \beta(t), \\ \phi_i^*(t, x) &= \begin{cases} \max \left\{ 0, \zeta_{i(t)}(t) \left( (\Psi(t) - 1)x + \Psi(t)A(t) \right) \right\} & , \text{if } i = i^*(t) \\ 0 & , \text{otherwise} , \end{cases} \\ q_j^*(t, x) &= \begin{cases} \max \left\{ 0, h_{j(t)}(t) \left( E(t) (x + A(t)) \right) \right\} & , \text{if } j = j^*(t) \\ 0 & , \text{otherwise.} , \end{cases}\end{aligned}$$

where

$$\begin{aligned}
A(t) &= \int_t^T m(s) e^{(-\int_t^s (r(v) + \zeta_{i^*}(v)) dv)} ds, \\
\Psi(t) &= \frac{1}{G(t)} \left( \frac{\xi(t)}{\zeta_{i^*}(t)} \right)^{1/(1-\gamma)}, \\
G(t) &= e^{-\int_t^T \Pi(u) du} + \int_t^T \Lambda(s) e^{-\int_t^s \Pi(u) du} ds, \\
\Pi(t) &= \frac{\xi(t) + \rho}{1-\gamma} - \frac{\gamma}{1-\gamma} (r(t) + \zeta_{i^*}(t)) - \frac{\gamma}{(1-\gamma)^2} \Gamma(t), \\
\Lambda(t) &= 1 + \left( \frac{\zeta_{i^*}^\gamma(t)}{\xi(t)} \right)^{\frac{1}{\gamma-1}} + \left( \frac{h_{j^*}^\gamma(t)}{\xi(t)} \right)^{\frac{1}{\gamma-1}}, \\
\Gamma(t) &= \beta^T(t) \xi \beta(t) - \frac{1}{2} \|\sigma^T \xi \beta(t)\|^2, \\
E(t) &= \frac{1}{G(t)} \left( \frac{\xi(t)}{h_{j^*}^\gamma(t)} \right)^{\frac{1}{1-\gamma}}.
\end{aligned}$$

*Proof.* Assume that the utility function are given as (3.25). From Theorem 4.1 we have the following condition

$$L_\kappa(t, \kappa) - V_x(t, x) = 0, \quad (4.11)$$

$$x^2 V_{xx}(t, x) \theta \sigma \sigma^T + (\mu(t) - r(t)) x V_x(t, x) = 0, \quad (4.12)$$

$$\frac{\xi(t)}{h_j^*(t)} Y_x \left( t, \frac{q_{j^*}(t)}{h_j^*(t)} \right) - V_x(t, x) = 0, \quad (4.13)$$

$$\frac{\xi(t)}{\zeta_i^*(t)} Y_x \left( t, x + \frac{\phi_{i^*}(t)}{\zeta_{i^*}(t)} \right) - V_x(t, x) = 0. \quad (4.14)$$

Differentiate  $L$  with respect to  $\kappa$

$$L_\kappa(t, \kappa) = e^{-\rho t} (\kappa^*(t, x))^{\gamma-1}.$$

Then substitute above equation in (4.11) we get

$$V_x(t, x) = e^{-\rho t} (\kappa^*(t, x))^{\gamma-1}.$$

Rearrange the above equation for  $\kappa$  we have

$$\kappa^*(t, x) = \left( e^{\rho t} V_x(t, x) \right)^{\frac{-1}{(1-\gamma)}}. \quad (4.15)$$

From equation (4.12)

$$\theta^*(t, x) = -\frac{V_x(t, x)}{xV_{xx}(t, x)}\varepsilon\beta(t). \quad (4.16)$$

To find the values of  $\phi$  and  $q$ . Differentiate  $Y$  with respect to  $x$  and the substitute it in (4.13) we obtain

$$e^{-\rho t} \left( x + \frac{\phi_{i^*}(t)}{\zeta_{i^*}(t)} \right)^{\gamma-1} = \frac{\phi_{i^*}(t)V_x(t, x)}{\xi(t)},$$

where

$$Y_x(t, \Upsilon) = e^{-\rho t}\Upsilon^{\gamma-1}.$$

Hence

$$\phi_i^*(t, x) = \begin{cases} \max \left\{ 0, \left( \left( \frac{\zeta_i(t)e^{\rho t}V_x(t, x)}{\xi(t)} \right)^{\frac{-1}{(1-\gamma)}} - x \right) \zeta_I(t) \right\}, & \text{if } i = i^*(t) \\ 0 & \text{otherwise.} \end{cases} \quad (4.17)$$

Now do the similar for  $q$

$$e^{-\rho t} \left( \frac{q_{l^*}(t)}{h_{l^*}(t)} \right)^{\gamma-1} = \frac{h_{l^*}(t)}{V_x(t, x)}\xi(t).$$

Consequently

$$q_j^*(t, x) = \begin{cases} \max \left\{ 0, \left( \left( \frac{h_j(t)e^{\rho t}V_x(t, x)}{\xi(t)} \right)^{\frac{-1}{(1-\gamma)}} \right) h_{j(t)} \right\}, & \text{if } j = j^*(t) \\ 0 & \text{otherwise.} \end{cases} \quad (4.18)$$

Substitute all above equations(4.15, 4.16, 4.17), and (4.18) in the HJB equation we get

$$\sup_{(\kappa, \phi, q, \theta) \in \mathbb{R} \times (\mathbb{R}_0^+)^I \times \mathbb{R}^L \times \mathbb{R}^N} \mathcal{H}(t, x, \kappa, \phi, q, \theta) = \frac{e^{-\rho t} \left( e^{\rho t} V_x(t, x) \right)^{\frac{\gamma}{\gamma-1}}}{\gamma}$$



$$\begin{aligned}
& - \left( e^{\rho t} V_x(t, x) \right)^{\frac{1}{\gamma-1}} V_x(t, x) + \frac{1}{2} \frac{V_x^2}{V_{xx}} \left\| \sigma^T \xi \beta(t) \right\|^2 - \frac{V_x^2}{V_{xx}} \beta^T \xi \beta \\
& + \frac{e^{-\rho t} \xi(t)}{\gamma} \left( \frac{\zeta_{i^*}(t) e^{\rho t} V_x(t, x)}{\xi(t)} \right)^{\frac{\gamma}{\gamma-1}} - V_x(t, x) \left( \left( \frac{\zeta_{i^*}(t) e^{\rho t} V_x(t, x)}{\xi(t)} \right)^{\frac{1}{\gamma-1}} - x \right) \zeta_{i^*}(t) \\
& + \xi(t) e^{-\rho t} \frac{\left( \left( \frac{h_{l^*}(t) e^{\rho t} V_x(t, x)}{\xi(t)} \right)^{\frac{1}{\gamma-1}} \right)^\gamma}{\gamma} - V_x(t, x) \left( \left( \frac{h_{l^*}(t) e^{\rho t} V_x(t, x)}{\xi(t)} \right)^{\frac{1}{\gamma-1}} \right) h_{l^*}(t).
\end{aligned}$$

Rearrange the above terms we get

$$\begin{aligned}
& \sup_{(\kappa, \phi, q, \theta) \in \mathbb{R} \times (\mathbb{R}_0^+)^I \times \mathbb{R}^L \times \mathbb{R}^N} \mathcal{H}(t, x, \kappa, \phi, q, \theta) \\
& = e^{\frac{\rho t}{\gamma-1}} \left( \frac{1-\gamma}{\gamma} \right) \left( V_x(t, x) \right)^{\frac{\gamma}{\gamma-1}} \left[ 1 + \frac{\left( \zeta_{i^*}(t) \right)^{\frac{\gamma}{\gamma-1}} + \left( h_{l^*}(t) \right)^{\frac{\gamma}{\gamma-1}}}{\left( \xi(t) \right)^{\frac{1}{\gamma-1}}} \right] \\
& + V_x(t, x) \left( m(t) + x(\zeta_{i^*}(t) + r(t)) \right) \\
& - \frac{V_x^2}{V_{xx}} \left( \beta^T \xi \beta - \frac{1}{2} \left\| \sigma^T \xi \beta(t) \right\|^2 \right).
\end{aligned}$$

Let

$$\Lambda = \left[ 1 + \frac{\left( \phi_{i^*}(t) \right)^{\frac{\gamma}{\gamma-1}} + \left( h_{l^*}(t) \right)^{\frac{\gamma}{\gamma-1}}}{\left( \xi(t) \right)^{\frac{1}{\gamma-1}}} \right].$$

$$\Gamma(t) = \beta^T(t) \xi \beta(t) - \frac{1}{2} \left\| \sigma^T \xi \beta(t) \right\|^2.$$

The HJB equation become

$$\begin{aligned}
& V_t(t, x) - \xi(t) V(t, x) + e^{\frac{\rho t}{\gamma-1}} \left( \frac{1-\gamma}{\gamma} \right) \left( V_x(t, x) \right)^{\frac{\gamma}{\gamma-1}} \Lambda \\
& + V_x(t, x) \left( m(t) + x(\zeta_{i^*}(t) + r(t)) \right) + \Gamma(t) \frac{V_x^2(t, x)}{V_{xx}(t, x)} = 0, \quad (4.19)
\end{aligned}$$

---

where  $\Lambda, \Gamma$  as given in the statement of the proposition with the terminal condition

$$V(T, x) = R(x). \quad (4.20)$$

In order to solve equation (4.19) we do the following steps:

- Consider the ansatz function as

$$V(t, x) = \frac{a(t)}{\gamma} (x + A(t))^\gamma.$$

- Find the derivatives of  $V_t$ ,  $V_x$  and  $V_{xx}$

$$\begin{aligned} V_t(t, x) &= a(t) (x + A(t))^{\gamma-1} \frac{dA(t)}{dt} + \frac{1}{\gamma} (x + A(t))^\gamma \frac{da(t)}{dt}, \\ V_x(t, x) &= a(t) (x + A(t))^{\gamma-1}, \\ V_{xx}(t, x) &= (\gamma - 1) a(t) (x + A(t))^{\gamma-2}. \end{aligned} \quad (4.21)$$

- Substitute above partial derivative in equation (4.19) to solve it, we get

$$\begin{aligned} & a(t) (x + A(t))^{\gamma-1} \frac{dA(t)}{dt} + \frac{1}{\gamma} (x + A(t))^\gamma \frac{da(t)}{dt} - \xi(t) \frac{a(t)}{\gamma} (x + A(t))^\gamma \\ & + e^{\frac{\rho t}{\gamma-1}} \left( \frac{1-\gamma}{\gamma} \right) \left( a(t) (x + A(t))^{\gamma-1} \right)^{\frac{\gamma}{\gamma-1}} \Lambda \\ & + a(t) (x + A(t))^{\gamma-1} \left( m(t) + x(\zeta_{u^*}(t) + r(t)) \right) \\ & + \Gamma(t) \frac{\left( a(t) (x + A(t))^{\gamma-1} \right)^2}{(\gamma - 1) a(t) (x + A(t))^{\gamma-2}} = 0. \end{aligned}$$

- 
- Multiply previous equation  $\left(x + A(t)\right)^{-\gamma}$  we obtain

$$\begin{aligned}
& \frac{a(t)}{x + A(t)} \frac{dA(t)}{dt} + \frac{1}{\gamma} \frac{da(t)}{dt} - \xi(t) \frac{a(t)}{\gamma} \\
& + e^{\frac{-\rho t}{1-\gamma}} \left(\frac{1-\gamma}{\gamma}\right) \left(a(t)\right)^{\frac{-\gamma}{1-\gamma}} \Lambda \\
& + \frac{a(t)}{x + A(t)} \left(m(t) + x(\zeta_{i^*}(t) + r(t))\right) \\
& + \Gamma(t) \frac{a(t)}{\gamma - 1} = 0.
\end{aligned}$$

- Add  $\frac{r(t)a(t)A(t)}{x+A(t)}$  and  $\frac{\zeta_{i^*}(t)a(t)A(t)}{x+A(t)}$  to both sides we get

$$\begin{aligned}
& \frac{1}{\gamma} \frac{da(t)}{dt} - \xi(t) \frac{a(t)}{\gamma} + \frac{a(t)}{x + A(t)} \frac{dA(t)}{dt} + \frac{m(t)a(t)}{x + A(t)} \\
& + \frac{xr(t)a(t)}{x + A(t)} + \frac{r(t)a(t)A(t)}{x + A(t)} + \frac{x\zeta_{i^*}(t)a(t)}{x + A(t)} \\
& + \frac{\zeta_{i^*}(t)a(t)A(t)}{x + A(t)} + e^{\frac{-\rho t}{1-\gamma}} \left(\frac{1-\gamma}{\gamma}\right) \left(a(t)\right)^{\frac{-\gamma}{1-\gamma}} \Lambda + \Gamma(t) \frac{a(t)}{\gamma - 1} \\
& = \frac{r(t)a(t)A(t)}{x + A(t)} + \frac{\zeta_{i^*}(t)a(t)A(t)}{x + A(t)}.
\end{aligned}$$

- To solve previous equation we separated it in two independent boundary value problem one for  $a(t)$  and other for  $A(t)$

$$\begin{aligned}
& \frac{1}{\gamma} \frac{da(t)}{dt} - \xi(t) \frac{a(t)}{\gamma} + \Gamma(t) \frac{a(t)}{\gamma - 1} + e^{\frac{-\rho t}{1-\gamma}} \left(\frac{1-\gamma}{\gamma}\right) \left(a(t)\right)^{\frac{-\gamma}{1-\gamma}} \Lambda \\
& + \zeta_{i^*}(t)a(t) + r(t)a(t) = 0.
\end{aligned}$$

Rearrange above equation we obtain

$$\left(\zeta_{i^*}(t) + r(t) + \frac{\Gamma(t)}{\gamma - 1} - \frac{\xi(t)}{\gamma}\right) a(t) + e^{\frac{-\rho t}{1-\gamma}} \left(\frac{1-\gamma}{\gamma}\right) \left(a(t)\right)^{\frac{-\gamma}{1-\gamma}} \Lambda + \frac{1}{\gamma} \frac{da(t)}{dt} = 0,$$

$$a(T) = e^{-\rho T}. \tag{4.22}$$

---

And for A

$$\frac{a(t)}{x + A(t)} \frac{dA(t)}{dt} + \frac{m(t)a(t)}{x + A(t)} - \frac{r(t)a(t)A(t)}{x + A(t)} - \frac{\zeta_{i^*}(t)a(t)A(t)}{x + A(t)} = 0.$$

Multiply above equation we get

$$\begin{aligned} \frac{dA(t)}{dt} + A(t) \left( -r(t) - \zeta_{i^*}(t) \right) + m(t) &= 0, \\ A(T) &= 0. \end{aligned} \quad (4.23)$$

- Let us start solving equation (4.23) rewrite previous equation as:

$$\frac{dA(t)}{dt} + A(t) \left( -r(t) - \zeta_{i^*}(t) \right) = -m(t).$$

The above equation is first order linear differential equation which can be solve by integrating factor and the solution is in the form:

$$A(t) = \frac{1}{\mu(t)} \left( \int_t^T \mu(s) (-m(s)) ds + C \right),$$

where

$$\mu(t) = e^{-\int_t^s [r(u) + \zeta(u)] du}.$$

Hence,

$$A(t) = e^{\int_t^s [r(u) + \zeta(u)] du} \left( \int_t^T e^{-\int_t^s [r(u) + \zeta(u)] du} (-m(s)) ds + C \right).$$

Hence,

$$A(t) = e^{\int_t^s [r(u) + \zeta_{i^*}(u)] du} \left( \int_t^T e^{\int_t^s [-r(u) - \zeta_{i^*}(u)] du} (-m(s)) ds + C \right).$$

Using boundary condition we get:

$$A(t) = \int_t^T m(s) e^{-\int_t^s (r(u) + \zeta_{i^*}(u)) du} ds.$$

- To solve equation (4.22) we assume that the solution is on the form:

$$\begin{aligned} a(t) &= e^{-\rho t}(G(t))^{1-\gamma}, \\ a(T) &= e^{-\rho T}. \end{aligned} \tag{4.24}$$

Differentiate  $a(t)$  with respect to time  $t$ , we obtain;

$$\frac{da(t)}{dt} = e^{-\rho t}(1-\gamma)(G(t))^{-\gamma} \frac{dG(t)}{dt} + (G(t))^{1-\gamma} (-\rho e^{-\rho t}).$$

Substitute the derivative of  $a(t)$  in equation (4.22) we obtain:

$$\begin{aligned} &\frac{1}{\gamma} \left( e^{-\rho t}(1-\gamma)(G(t))^{-\gamma} \frac{dG(t)}{dt} - \rho e^{-\rho t}(G(t))^{1-\gamma} \right) \\ &+ \left( \zeta_{i^*}(t) + r(t) + \frac{\Gamma(t)}{\gamma-1} - \frac{\xi(t)}{\gamma} \right) e^{-\rho t}(G(t))^{1-\gamma} \\ &+ e^{\frac{-\rho t}{1-\gamma}} \left( \frac{1-\gamma}{\gamma} \right) \left( e^{-\rho t}(G(t))^{1-\gamma} \right)^{\frac{-\gamma}{1-\gamma}} \Lambda = 0. \end{aligned}$$

Multiply the previous equation by  $\frac{\gamma}{1-\gamma}$ , we get;

$$\begin{aligned} &e^{-\rho t}(G(t))^{-\gamma} \frac{dG(t)}{dt} + (G(t))^{1-\gamma} \left( \frac{-\rho}{1-\gamma} e^{-\rho t} \right) \\ &\frac{\gamma}{1-\gamma} \left( \zeta_{k^*}(t) + r(t) + \frac{\Gamma(t)}{\gamma-1} - \frac{\xi(t)}{\gamma} \right) e^{-\rho t}(G(t))^{1-\gamma} \\ &+ e^{-\rho t} G(t)^{-\gamma} \Lambda(t) = 0. \end{aligned}$$

Divide the above equation by  $e^{-\rho t}$  we obtain:

$$\begin{aligned} &G(t)^{-\gamma} \frac{dG(t)}{dt} + (G(t))^{1-\gamma} \frac{-\rho}{1-\gamma} + G(t)^{-\gamma} \Lambda(t) \\ &\frac{\gamma}{1-\gamma} \left( \zeta_{k^*}(t) + r(t) + \frac{\Gamma(t)}{\gamma-1} - \frac{\xi(t)}{\gamma} \right) e^{-\rho t}(G(t))^{1-\gamma} = 0. \end{aligned}$$

---

Divide by  $G(t)^{-\gamma}$ , we get:

$$\frac{dG(t)}{dt} + G(t)^{\frac{-\rho}{1-\gamma}} + \Lambda(t) \frac{\gamma}{1-\gamma} \left( \zeta_{k^*}(t) + r(t) + \frac{\Gamma(t)}{\gamma-1} - \frac{\xi(t)}{\gamma} \right) e^{-\rho t} G(t) = 0.$$

Rearrange above equation:

$$\frac{dG(t)}{dt} + \left( \frac{\xi(t) + \rho}{1-\gamma} - \frac{1}{2}\gamma \left( \frac{\mu(t) - r(t)}{(1-\gamma)\sigma(t)} \right)^2 - \frac{\gamma}{1-\gamma} (r(t) + \zeta_{i^*}(t)) \right) G(t) + \Lambda(t) = 0. \quad (4.25)$$

For simplify let:

$$\Pi(t) \triangleq \frac{\xi(t) + \rho}{1-\gamma} - \frac{1}{2}\gamma \left( \frac{\mu(t) - r(t)}{(1-\gamma)\sigma(t)} \right)^2 - \frac{\gamma}{1-\gamma} (r(t) + \zeta_{i^*}(t)), \quad (4.26)$$

Equation (3.40) can be rewritten as:

$$\begin{aligned} \frac{dG(t)}{dt} - \Pi(t)G(t) + \Lambda(t) &= 0, \\ G(T) &= 1. \end{aligned}$$

For simplify let:

$$\Pi(t) \triangleq \frac{\xi(t) + \rho}{1-\gamma} - \frac{1}{2}\gamma \left( \frac{\mu(t) - r(t)}{(1-\gamma)\sigma(t)} \right)^2 - \frac{\gamma}{1-\gamma} (r(t) + \zeta_{i^*}(t)), \quad (4.27)$$

Equation (4.25) can be rewritten as:

$$\begin{aligned} \frac{dG(t)}{dt} + \Pi(t)G(t) + \Lambda(t) &= 0, \\ G(T) &= 1. \end{aligned}$$

Above equation is linear first order differential equation and its solution given by:

$$G(t) = e^{\int_T^t \Pi(u)du} \left( \int_T^t -\Lambda(s) e^{\int_T^s -\Pi(u)du} ds + c \right).$$

---

From  $G(T) = 1$  we have:

$$G(t) = e^{-\int_t^T \Pi(u)du} + \int_t^T \Lambda(s) e^{\int_s^T -\Pi(u)du} e^{\int_t^s \Pi(u)du} ds.$$

Thus

$$G(t) = e^{-\int_t^T \Pi(u)du} + \int_t^T \Lambda(s) e^{-\int_t^s \Pi(u)du} ds. \quad (4.28)$$

Finally :

$$a(t) = e^{-\rho t} \left( e^{-\int_t^T \Pi(u)du} + \int_t^T \Lambda(s) e^{-\int_t^s \Pi(u)du} ds \right)^{1-\gamma}. \quad (4.29)$$

- Substitute the value of  $V_x$  from (4.21) in (4.15) we obtain:

$$\kappa^*(t, x) = \frac{1}{G(t)}(x + A(t)).$$

- Substitute the value of  $V_x$  and  $V_{xx}$  from (4.21) in (4.16) we obtain:

$$\theta^*(t, x) = \frac{\xi\beta(t)(x + A(t))}{x(1 - \gamma)}.$$

- For  $\phi$ , Let us consider the case where  $i = u^*$  and substitute the value of  $V_x$  we get:

$$\phi_I^*(t, x) = \left( \frac{\left( \zeta_{u^*}(t) \right)^{\frac{1}{\gamma-1}} (G(t))^{-1} (x + A(t))}{\left( \xi(t) \right)^{\frac{1}{\gamma-1}}} - x \right) \zeta_{u^*}(t).$$

Simplify it by assume that:

$$\Psi(t) = \frac{1}{G(t)} \left( \frac{\zeta_{i^*}(t)}{\xi(t)} \right)^{\frac{1}{\gamma-1}}.$$

So we have:

$$\phi^*(t, x) = \begin{cases} \max \left\{ 0, \left[ Z_2 \left( t, \frac{\zeta_I(t)V_x(t,x)}{\xi(t)} \right) - x \right] \zeta_I(t) \right\}, & \text{if } u = u^*(t) \\ 0 & \text{otherwise,} \end{cases}$$

where

$$i^*(t) = \arg \min_{i \in \{1, 2, \dots, I\}} \{\zeta_I(t)\}. \quad (4.30)$$

- Similarly, we can find  $q$  by assuming the case where  $j = j^*$  and substitute the value of  $V_x$  we get:

$$q^*(t, x) = \left( \frac{\left( h_{j^*}(t) \right)^{\frac{1}{\gamma-1}} (G(t))^{-1} (x + A(t))}{\left( \xi(t) \right)^{\frac{1}{\gamma-1}}} \right) h_j(t).$$

Simplify it by assume that:

$$E(t) = \frac{1}{G(t)} \left( \frac{\xi(t)}{h_{j^*}^\gamma(t)} \right)^{\frac{1}{1-\gamma}}.$$

Thus

$$q_j^*(t, x) = \begin{cases} \max \left\{ 0, h_{j(t)}(t) \left( E(t) (x + A(t)) \right) \right\} & , \text{if } j = j^*(t) \\ 0 & , \text{otherwise,} \end{cases}$$

Hence, we conclude the proof. □

**Remark 4.1.** (1) It seen from previous Proposition 4.1 that  $V(t, x)$ ,  $\kappa^*(t, x)$ ,  $\phi_u^*(t, x)$  and  $q_j^*(t, x)$  are all increasing with wealth  $x$ . However, the optimal risky-asset allocation  $\theta^*(t, x)$  is decreasing with wealth  $x$ .

(2) It seen from previous Proposition 4.1 that as  $t$  goes to  $T$ , we have that  $G(t) \rightarrow 1$  and  $A(t) \rightarrow 0$ . Hence, we observe that for an economic agent with a large wealth that is close to reaching pension age, the optimal social welfare purchase will tend to this limiting quantity of purchase

$$\left( \frac{\xi(t)}{h_{j^*}^\gamma(t)} \right)^{\frac{1}{1-\gamma}} x.$$



## Chapter 5

---

### Conclusion

---

*We have extended the work done by Mousa et al. [25] and Moath [20] by allowing the economic agent to contribute in a social welfare market in order to protect her family in the case of premature death.*

*We have studied the problem faced by the economic agent who is investing in the financial market composed of one risk-free asset and only one risky security, and have access to the life insurance market and social welfare policy consists of  $L$  providers. We have used dynamic programming technique to derive a second order nonlinear partial differential equation in order to maximize the expected utility. Finally, We have characterized the optimal strategies concerning consumption, investment, life-insurance selection and social welfare policy using family of discounted constant relative risk aversion utility functions.*

*In addition, we have introduced a generalized form for the economic agent to have access to the financial market comprised of  $N$  risky assets, life insurance market, and social welfare market composed of  $L$  providers. We used the idea of dynamic programming principle to get the optimal strategies. Also, We have determined the explicit solution in a special case of discounted constant relative risk aversion utility*

---

*functions. Finally, we have concluded some properties for the explicit solutions.*

---

## *Bibliography*

---

- [1] *R. P. Agarwal and D. O'Regan. An introduction to ordinary differential equations. Springer Science Business Media, New York, 2008.*
- [2] *Y. Zhu An. Uncertain optimal control with application to a portfolio selection model, cybernetics and systems. Cybernetics and Systems: An International Journal, 41(7):535–547, 2010.*
- [3] *S. Andersen. On the collective theory of risk in case of contagion between claims. Bull. Inst. Math. Appl, 12:275–279, 1957.*
- [4] *S. Asmussen. Risk theory in a markovian environment. Scand. Actuar. J, page 69–100, 1989.*
- [5] *R. Bellman. On the theory of dynamic programming. Proc. Natl. Acad. Sci. USA, 38(8):716–719, 1952.*
- [6] *R. Bellman. Dynamic programming and a new formalism in the calculus of variations. Proc. Natl. Acad. Sci. USA, 40:231–235, 1954.*
- [7] *R. Bellman. Dynamic programming and stochastic control process. Inf. Control, page 228–239, 1958.*

- 
- [8] V. Capasso and D. Bakstein. An introduction to continuous-time stochastic processes. *Birkhax user Boston*, 2005.
- [9] I. Duarte, D. Pinheiro, A. A. Pinto, and S. R. Pliska. *Optimal life insurance purchase, consumption and investment on a financial market with multi-dimensional diffusive terms*. *Optimization*, (11):1737–1760, 2012.
- [10] B. Dumas and E. Luciano. *An exact solution to a dynamic portfolio choice problem under transactions costs*. *J. Finance*, 46:577–595, 1991.
- [11] M. Elias and S. Shakarchi. *Measure theory, integration, and hilbert spaces*. *Princeton University, New Jersey*, 2005.
- [12] L.C. Evans. *An introduction to stochastic differential equations, volume 82*. *Am. Math. Soc, Providence, Rhode Island*, 2012.
- [13] S. Fischer. *A life cycle model of life insurance purchase*. *Internat. Eco. Rev*, page 132–152, 1973.
- [14] W. Fleming and R. Rishel. *Deterministic and stochastic optimal control*. *Springer-Verlag, Berlin*, 1975.
- [15] W. Fleming and H. Soner. *Controlled markov processes and viscosity solutions*. *Springer-Verlag, United States of America, second edition*, 2006.
- [16] B. George, D. Maurice, and R. Hass. *Thomas’ calculus*. *Pearson Education: Addison-Wesley*, 2005.
- [17] N. Hakansson. *Optimal investment and consumption strategies under risk, an uncertain lifetime, and insurance*. *Internat. Econom. Rev*, (14):132–152, 1969.
- [18] N. Hakansson. *Optimal entrepreneurial decisions in a completely stochastic environment*. *Manage. Sci*, (17):427–449, 1971.
- [19] A. Hanson. *On sufficiency of the kuhn-tucker conditions*. *La Trobe University, Bundoora 3083, Australia, pages 545–550*, 1981.

- 
- [20] M. Hoshiea. *Optimal strategies for a wage earner under controlibility of consumption, investment, industrial and welfare rates. Master's thesis, Birzeit University, 2022.*
- [21] S. Khalil. *Optimal strategies for an economic agent in a continuous time model within social security system. Master program in mathematics, Birzeit University, 2018.*
- [22] O. Knill. *Probability Theory and Stochastic Processes with Applications. Havard Web-Based, USA, 1994.*
- [23] R .C. Merton. *Lifetime portfolio selection under uncertainty: The continuous time case. Rev. Eco. Stat, (51):247–257, 1969.*
- [24] R. C. Merton. *Optimum consumption and portfolio rules in a continuous-time model. J. Eco. Theory, (3):372–413, 1971.*
- [25] A. S. Mousa, D. Pinheiro, and A. Pinto. *Optimal life-insurance selection and purchase within a market of several life-insurance providers. Insurance Math. Eco, (67):133–144, 2016.*
- [26] H. Ou-Yang. *Optimal contracts in a continuous-time delegated portfolio management problem. Rev. Financ. Stud, 1(6):173–208, 2003.*
- [27] S. R. Pliska and J. Ye. *Optimal life insurance purchase, consumption and investment. Finance Department, University of Chicago, 2007.*
- [28] H. Royden and P. Fitzpatrick. *Real analysis, volume 32. Macmillan, New York, 1988.*
- [29] B. Sendal. *Stochastic differential equations. Springer-Verlag Heidelberg, New York, fifth edition, 2003.*
- [30] Y. Shenab and J. Weib. *Optimal investment-consumption-insurance with random parameters. Scand. Actuar. J, 2014.*
- [31] L. Shepley. *Introduction to ordinary differential equations. Halliday, fourth edition, 1989.*

- 
- [32] H. Mete Soner and N. Touzi. *Homogenization and asymptotics for small transaction costs*. SIAM J, 2018.
- [33] S. Timo. *Basics of stochastic analysis*. University of Wisconsin–Madison, 2012.
- [34] M .E. Yaari. *Uncertain lifetime, life insurance, and the theory of the consumer*. Rev. Econom. Stud., (32):137–150, 1965.
- [35] J. Ye. *Optimal Life Insurance Purchase, Consumption and Portfolio under an Uncertain Life*. PhD thesis, University of Illinois at Chicago, Chicago, 2006.
- [36] J. Ye. *Optimal life insurance, consumption and portfolio under uncertainty*. Martingale methods. the 26th American Control Conference, 2007.
- [37] J. Ye. *Optimal life insurance purchase, consumption and investment stochastic utilities with subsistence and satiation*. Insurance Math. Eco, 2019.
- [38] J. Ye. *Random distribution kernels and three types of defaultable contingent payoffs*. Insurance Math. Eco, 2019.
- [39] J. Yong and X. Zhou. *Stochastic controls: hamiltonian systems and hjb. equations*. Springer-Verlag, 1999.
- [40] X. Y. Zhou, X. Li, and A. E. Lim. *Dynamic mean-variance portofolio selection with no-shorting constraint*. SIAM J, 40(5):1540–1555, 2002.